Introduction of ϕ-Geometry

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March 1, 2017

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Let M denote a smooth m-dimensional manifold with non-empty boundary $N := \partial M$ and write $\mathring{M} := M \setminus \partial M$ for its interior. Furthermore let there be a local trivial fibre bundle $(N, \phi, B; \mathcal{F})$ where the total space N, the base space B and the fibre type $\mathcal{F}$ are manifolds and the projection $\phi\colon N \to B$ is a surjective submersion.

In this document we want to introduce the basic notion of the ϕ-tangent bundle and classes of bundle metrics on it. While it is possible to only describe these ϕ-metrics by their behaviour on the interior $\mathring{M}$, it is their specific singular behaviour on ∂M which characterizes them. This is best introduced using the abstract ϕ-tangent bundle.

The ϕ-tangent bundle was introduced in [MM98] and is modelled on the b-tangent bundle as defined in [Mel93, Section 2.2.]. Note that its construction depends on the fibre bundle ϕ and partially on the choice of a boundary defining function x. Compare the paragraph after [MM98, Lemma 3].

1 Definition: "Boundary Defining Function"

A function $x\colon M \to \mathbb{R}$ is called *boundary defining*, if

$$x(p) = 0 \qquad \text{and} \qquad \mathrm{d}x_p \neq 0 \tag{1.1}$$

on the boundary $\partial M \ni p$.

2 Definition: "ϕ-Tangent Bundle"

Let x be a boundary defining function of M and let there be a locally trivial fibre bundle $(N, \phi, B; \mathcal{F})$. Then we can define the *ϕ-vector fields* as

$${}^{\phi}\mathcal{V}(M) := \left\{ X \in \mathcal{V}(M) \;\middle|\; X|_N \in \ker \mathrm{d}\phi, Xx \in \mathcal{O}(x^2) \right\}, \tag{2.1}$$

i.e. those that vanish in horizontal direction of first order and in normal direction of second order. Now we construct the abstract *ϕ-tangent bundle*, whose sections these are, up to isomorphism. For $p \in M$ we define the quotient vector space

$${}^{\phi}T_pM := {}^{\phi}\mathcal{V}(M) \big/ \mathcal{I}_p \cdot {}^{\phi}\mathcal{V}(M) \qquad \text{with} \qquad \mathcal{I}_p := \{ f \in C^\infty(M) \mid f(p) = 0 \}, \tag{2.2}$$

where $\mathcal{I}_p \cdot {}^{\phi}\mathcal{V}(M)$ contains finite sums of products fX, $f \in \mathcal{I}_p$, $X \in {}^{\phi}\mathcal{V}(M)$. Write $[X]_p := X + \mathcal{I}_p \cdot {}^{\phi}\mathcal{V}(M)$ for the projection ${}^{\phi}\mathcal{V}(M) \to {}^{\phi}T_pM$.

The method employed in this definition is called *rescaling* of a vector bundle. Also compare [MM98, Lemma 2] and [Mel93, Equation (8.4)].

3 Lemma: Vector Bundle Structure

${}^{\phi}TM$ possesses a vector bundle structure and over $\mathring{M}$ it is naturally isomorphic to TM.

Proof

Local Definition First of all, take $X, Y \in {}^{\phi}\mathcal{V}(M)$ and $p \in M$. If there is an open neighbourhood $U \ni p$ such that $X|_U = Y|_U$, choose a smooth cut-off function $f \in \mathcal{I}_p$ with $f|_{M\setminus U} = 1$. Then because $(X - Y)|_U = 0$, we get $X - Y = f(X - Y) \in \mathcal{I}_p \cdot {}^{\phi}\mathcal{V}(M)$. This shows, that the elements in ${}^{\phi}T_pM$ only depend on the germ at p of their representative in ${}^{\phi}\mathcal{V}(M)$.

Interior Now over the interior $\mathring{M}$ the restrictions of ${}^\phi\mathcal{V}(M)$ do not apply, i.e. ${}^\phi\mathcal{V}(M)|_{\mathring{M}} = \mathcal{V}(M)|_{\mathring{M}}$. In particular with $p \in \mathring{M}$ for every germ at p of some $X \in \mathcal{V}(M)$ there is a $Y \in {}^\phi\mathcal{V}(M)$ representing the same germ. This means, together with the previous result, that

$$ {}^\phi T_pM \simeq \mathcal{V}(M)\big/\mathcal{I}_p \cdot \mathcal{V}(M). \tag{3.1}$$

The fact that TM is a vector bundle allows us to choose a local frame $(e_1, \ldots, e_m) \in \mathrm{GL}(M)|_U$ over some $U \ni p$ so that every $X \in \mathcal{V}(M)$ has a unique decomposition $X|_U = \sum_i f_i e_i$, $f_i \in C^\infty(U)$. Immediately we see that $X(p) = 0 \Leftrightarrow \forall i\colon f_i(p) = 0$, which implies $\mathcal{I}_p \cdot \mathcal{V}(M) = \{\, X \in \mathcal{V}(M) \mid X(p) = 0 \,\}$. So then we have

$$ {}^\phi T_pM \simeq \mathcal{V}(M)\big/\{\, X \in \mathcal{V}(M) \mid X(p) = 0 \,\} \simeq T_pM \tag{3.2}$$

and the vector bundle structure of ${}^\phi TM$ over $\mathring{M}$ is just that of TM. The isomorphism is given explicitly by the evaluation map $\mathrm{ev}_p\colon \mathcal{V}(M) \ni X \mapsto X(p) \in TM$. On the boundary N however ev_p maps into the vertical subbundle $T^V N := \ker \mathrm{d}\phi$ consisting of the tangent bundles of the fibres $\phi^{-1}(q), q \in B$ and can not be an isomorphism.

Boundary Frames The boundary defining function x defines an extension $\overrightarrow{TN} := \ker \mathrm{d}x$ of TN on a neighbourhood of N. For $q \in N$ take any local frame $e := (T, e^H_1, \ldots, e^H_{\dim B}, e^V_1, \ldots, e^V_{\dim \mathcal{F}})$ of TM with $(e^V_1, \ldots, e^V_{\dim \mathcal{F}})$ spanning $T^V N$ and $(e^H_1, \ldots, e^H_{\dim B}, e^V_1, \ldots, e^V_{\dim \mathcal{F}})$ spanning $\overrightarrow{TN}$ over some open neighbourhood $U \subset M$ of q small enough that $x|_{U \setminus N} > 0$. Then in this frame ϕ-vector fields take the form

$$ X = f x^2 T + \sum_i f^H_i x e^H_i + \sum_j f^V_j e^V_j \quad \text{with} \quad f, f^H_i, f^V_j \in C^\infty(M). \tag{3.3}$$

Thus ${}^\phi\mathcal{V}(M)$ is spanned over $C^\infty(M)$ by ${}^\phi e := (x^2 T, x e^H_1, \ldots, x e^H_{\dim B}, e^V_1, \ldots, e^V_{\dim \mathcal{F}})$ on U. For $p \in U \setminus N$ the vectors ${}^\phi e_p$ remain a basis of T_pM and so we already know that $[{}^\phi e]_p$ is a basis of ${}^\phi T_pM$. For $p \in U \cap N$ define $\tilde{f} := f - f(p) \in \mathcal{I}_p$, $\tilde{f}^V_j := f^V_j - f^V_j(p) \in \mathcal{I}_p$ and so on. Then

$$ X = f(p)x^2 T + \sum_i f^H_i(p) x e^H_i + \sum_j f^V_j(p) e^V_j + \underbrace{\tilde{f} x^2 T + \sum_i \tilde{f}^H_i x e^H_i + \sum_j \tilde{f}^V_j e^V_j}_{\in \mathcal{I}_p \cdot {}^\phi\mathcal{V}(M)} . \tag{3.4}$$

So $[{}^\phi e]_p$ spans ${}^\phi T_pM$ over $\mathbb{R}$.

For $X \in \mathcal{I}_p \cdot {}^\phi\mathcal{V}(M)$ consider only the restriction onto a sufficiently short curve c starting at p in a direction outside of TN. Then c can be parametrized by x and all functions in $\mathcal{I}_p$ are of the form $x\varphi$, $\varphi \in C^\infty(M)$ over c. So over c we get the following representation for X

$$ \begin{aligned} X &= \sum_k x\varphi_k \Big(f_k x^2 T + \sum_i f^H_{k,i} x e^H_i + \sum_j f^V_{k,j} e^V_j\Big) \\ &= \Big(x \sum_k \varphi_k f_k\Big) x^2 T + \sum_i \Big(x \sum_k \varphi_k f^H_{k,i}\Big) x e^H_i + \sum_j \Big(x \sum_k \varphi_k f^V_{k,j}\Big) e^V_j . \end{aligned} \tag{3.5}$$

To be in $\operatorname{span}_{\mathbb{R}} {}^{\phi}e$ the coefficients have to be constant and with the formula for X above that is only possible if they vanish, i.e. $\sum_k \varphi_k f_k = 0$ and so on. This shows that ${}^{\phi}e$, apart from being linearly independent, is also complementary to $\mathcal{I}_p \cdot {}^{\phi}\mathcal{V}(M)$ and so $[{}^{\phi}e]_p$ is a basis of ${}^{\phi}T_pM$ for all $p \in U$.

Boundary Cocycles Now for e and any other local frame $\tilde{e}$ of this type, we have the cocycle $G \in C^{\infty}(U, \mathrm{GL}(m, \mathbb{R}))$, mapping e to $\tilde{e}$, which is a base transformation at every point. Applying this transformation to ${}^{\phi}e$ gives

$$G(x^2T) = x^2G(T) = G^{NN}x^2T + \sum_i xG_i^{NH}xe_i^H + \sum_j x^2G_j^{NV}e_j^V \tag{3.6}$$

$$G(xe_k^H) = xG(e_k^H) = G_k^{HN}xT + \sum_i G_{ki}^{HH}xe_i^H + \sum_j xG_{kj}^{HV}e_j^V \tag{3.7}$$

$$G(e_l^V) = G_l^{VN}T + \sum_i G_{li}^{VH}e_i^H + \sum_j G_{lj}^{VV}e_j^V. \tag{3.8}$$

As G has to preserve the subbundles $\overrightarrow{TN}$ and T^VN, we get $G_k^{HN} = G_l^{VN} = 0$ and $G_{li}^{VH} \in \mathcal{O}(x)$. Thus we have found the cocycle ${}^{\phi}G$ of ${}^{\phi}TM$ transforming ${}^{\phi}e$ into ${}^{\phi}\tilde{e}$ as

$$\begin{pmatrix} G^{NN} & 0 & 0 \\ xG_i^{NH} & G_{ki}^{HH} & {}^{\phi}G_{li}^{VH} \\ x^2G_j^{NV} & xG_{kj}^{HV} & G_{lj}^{VV} \end{pmatrix} \tag{3.9}$$

with $G_{li}^{VH} = x\,{}^{\phi}G_{li}^{VH}$. The ${}^{\phi}G_{li}^{VH}$ are smooth, so ${}^{\phi}G$ is smooth as well.

4 Theorem: Collar Neighbourhood

There is a *collar neighbourhood* (or *collar*) of ∂M, i.e. an open neighbourhood $U \subset M$ of ∂M which is diffeomorphic to $[0, C) \times \partial M$ with $C \in \mathbb{R}, C > 0$ and ∂M being associated to $\{0\} \times \partial M$.

Proof The theorem is an analogon of the tubular neighbourhood theorem. See [Spi99, Chapter 9 Theorem 20. and 21.] for a proof in the case where ∂M is compact. That is the case which concerns us, although the statement remains true without that assumption.

5 Remark:

If $U \subset M$ is a collar of ∂M with diffeomorphism $\Psi\colon [0, C) \times \partial M \to U$, then there is a unique associated boundary defining function $x\colon M \to \mathbb{R}$ such that $x(\Psi(x_0, p)) = x_0$.

On the other hand, given a boundary defining function x, we can always find a collar isomorphism Ψ *compatible* to x in the above sense that $x(\Psi(x_0, p)) = x_0$, for example by taking any collar isomorphism $\tilde{\Psi}$ and defining $\Psi^{-1}(\tilde{\Psi}(x_0, p)) \coloneqq (x(\tilde{\Psi}(x_0, p)), p)$.

6 Lemma: Global Isomorphism

${}^{\phi}TM$ is isomorphic to TM over all of M.

Proof Choose a collar isomorphism $\Psi\colon [0,1) \times N \to U \subset M$ compatible with x as well as a complement $T^H N$ of $T^V N$ in TN. Modify x smoothly outside of $\Psi([0,\frac{1}{3}) \times N)$ so that it is constant 1 outside of $\Psi([0,\frac{2}{3}) \times N)$. Then the decomposition $(\mathbb{R} \times [0,1) \times N) \oplus (T^H N \times [0,1)) \oplus (T^V N \times [0,1))$ is mapped to $TM|_U$ via $\mathrm{d}\Psi$. Furthermore for $X \in \mathcal{V}(M)$ decomposing over U as $X = X^N + X^H + X^V$ define

$$\mu(X)_p = \begin{cases} x^2 X_p^N + x X_p^H + X_p^V & \text{if } p \in U \text{ and} \\ X_p & \text{otherwise.} \end{cases} \tag{6.1}$$

Every ϕ-vector field has a unique representation as $\mu(X)$ and so $\mu\colon \mathcal{V}(M) \to {}^\phi\mathcal{V}(M)$ is bijective. It is also linear and $C^\infty(M)$-equivariant, thus $\mu(\mathcal{I}_p \cdot \mathcal{V}(M)) = \mathcal{I}_p \cdot \mu(\mathcal{V}(M)) = \mathcal{I}_p \cdot {}^\phi\mathcal{V}(M)$. That means μ induces an $\mathbb{R}$-linear bijection $\mu_p\colon T_pM \to {}^\phi T_p M$ for every p.

On $\mathring{M}$ we can take the same local frame for TM and ${}^\phi TM$. Then μ is multiplication of the coefficients by smooth, non-vanishing functions, so it is a diffeomorphism there. Around $p \in U$ we can take frames e and ${}^\phi e$ as in the previous proof and subordinate to the decomposition of TM. Then μ is the identity on the coefficients, so it is a diffeomorphism everywhere.

7 Lemma: Lie Algebra

The ϕ-vector fields ${}^\phi\mathcal{V}(M)$ form a Lie algebra. Thus we get an exterior derivative $\mathrm{d}\colon {}^\phi\Omega(M) \to {}^\phi\Omega(M)$, where ${}^\phi\Omega(M) := \Gamma(M, \Lambda^\phi T^* M)$.

Proof Take any $X, Y \in {}^\phi\mathcal{V}(M)$. By definition $X|_N$ and $Y|_N$ are tangent to N and ϕ-related to $0 \in \mathcal{V}(B)$ and so as is $[X,Y]|_N = [X|_N, Y|_N] \in \ker \mathrm{d}\phi$. Furthermore $Xx, Yx \in \mathcal{O}(x^2)$ means there are $f, g \in C^\infty(M)$ such that $Xx = x^2 f$ and $Yx = x^2 g$. Then calculate

$$\begin{aligned} [X,Y]x &= X(Yx) - Y(Xx) = X(x^2 g) - Y(x^2 f) \\ &= 2x(Xx)g + x^2 Xg - 2x(Yx)f - x^2 Yf \\ &= 2xfg + x^2 Xg - 2xgf - x^2 Yf = x^2(Xg - Yf), \end{aligned} \tag{7.1}$$

which shows that $[X,Y]x \in \mathcal{O}(x^2)$ and so $[X,Y] \in {}^\phi\mathcal{V}(M)$. Now d can be defined in the usual way through the Palais formula

$$\begin{aligned} (\mathrm{d}\alpha)(X_1, \dots, X_k) &= \sum_i (-1)^i X_i(\alpha(X_1, \dots, \cancel{X_i}, \dots, X_k)) \\ &\quad + \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_1, \dots, \cancel{X_i}, \dots, \cancel{X_j}, \dots, X_k) \end{aligned} \tag{7.2}$$

for $\alpha \in {}^\phi\Omega^k(M)$ and $X_i \in {}^\phi\mathcal{V}(M)$.

8 Definition: "ϕ-Metric"

A *ϕ-metric* is a positive-definite, smooth bundle metric on ${}^\phi TM$.

9 Definition: "ϕ-Connection"

A *ϕ-connection* on a vector bundle E, most importantly ${}^\phi TM$, is a linear map

$${}^\phi\nabla\colon \Gamma(E) \to \Gamma({}^\phi T^* M \otimes E) \tag{9.1}$$

that satisfies the usual Leibniz rule on its domain.

Given a ϕ-metric ${}^{\phi}g$ on M we can define a Levi-Civita ϕ-connection ${}^{\phi}\nabla^{LC}$ as usual via the Koszul formula, because the sections of ${}^{\phi}TM$ form a Lie algebra.

10 Definition: "Product and Exact ϕ-Metric"

A ϕ-metric ${}^{\phi}g$ on M is called a *product ϕ-metric*, if there is a collar neighbourhood $U \subset M$ of N with diffeomorphism $\Psi\colon [0, C) \times N \to U$, that is compatible with the boundary defining function x, and a connection $T^H N \subset TN$ such that ${}^{\phi}g$ takes the form

$$\Psi^*({}^{\phi}g) = \frac{(\mathrm{d}x)^2}{x_0^4} + \frac{\phi^* g_B}{x_0^2} + g_V, \quad \text{on } \{x_0\} \times N, x_0 \in [0, C), \tag{10.1}$$

where g_B is a metric on B and g_V is a metric on $T^V N = \ker \mathrm{d}\phi$, which both do not depend on x.

A ϕ-metric, which is a product ϕ-metric up to first order terms, i.e. differs from a product ϕ-metric by a symmetric bilinear form on ${}^{\phi}TM$ which vanishes over N, is called an *exact ϕ-metric*. Compare [Mor06, Definition 5] and the more abstract account in [Vai01, Section 1.2]. In contrast [LMP06] restrict the class of exact ϕ-metrics more and only allow variations in the direction of the hypersurfaces $x^{-1}(x_0)$.

11 Remark: Splitting on the Boundary

From the definition of ${}^{\phi}\mathcal{V}(M)$ we see, that the evaluation maps $\mathrm{ev}_p\colon {}^{\phi}\mathcal{V}(M) \to T_pM$ lift to give $\mathrm{Ev}_p\colon {}^{\phi}T_pM \to T_pM$ and as mentioned before $\operatorname{im} \mathrm{Ev}_p = T_p^V N$ for $p \in N$. So we get an invariantly defined subbundle $\ker \mathrm{Ev} \subset {}^{\phi}TM|_N$ of dimension

$$\dim {}^{\phi}TM - \dim \mathcal{F} = \dim TM - \dim \mathcal{F} = \dim B. \tag{11.1}$$

In addition we can conclude from (3.9), that the cocycles of ${}^{\phi}TM$ over N take the form

$$\begin{pmatrix} G^{NN} & 0 & 0 \\ 0 & G^{HH}_{ki} & {}^{\phi}G^{VH}_{li} \\ 0 & 0 & G^{VV}_{lj} \end{pmatrix} \tag{11.2}$$

so that, given a local frame ${}^{\phi}e = (x^2T, xe^H_1, \dots, xe^H_{\dim B}, e^V_1, \dots, e^V_{\dim \mathcal{F}})$ as before, the subbundles $\operatorname{span}\{x^2T\}$ and $\operatorname{span}\{xe^H_1, \dots, xe^H_{\dim B}\}$ of ${}^{\phi}TM|_N$ are independent of the choice of ${}^{\phi}e$. In contrast, there is no invariantly defined subbundle of vertical vectors.

In the above definition we extend the fibre bundle ϕ as well as the fixed connection $T^H N$ to a collar neighbourhood of N. This means for subordinate frames we get $G^{VH}\big|_U = 0$, which implies ${}^{\phi}G^{VH} = 0$ and thus gives us the subbundle of asymptotically vertical vectors in ${}^{\phi}TM$.

But as we can see in [Vai01, Section 1.2], to define exact ϕ-metrics, it would suffice to choose a decomposition of ${}^{\phi}TM$ as in [Vai01, Equation (10)], which corresponds to ${}^{\phi}G^{VH} \in \mathcal{O}(x) \Leftrightarrow G^{VH} \in \mathcal{O}(x^2)$.

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