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Universal Vector Bundles

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Contents

XeTeX The code was written in gvim with the help of vim-latex and compiled with xelatex, all on Gentoo Linux. Thanks to the free software community and to the fellow TeX users on TeX.SX for their great help and advice.
If you want to make any comments or suggest corrections to this script, please contact me! You can either email me directly or use the contact form on my website.

In the following we will always talk about *locally trivial* (vector/fibre) bundles.

1 Graßmann Manifolds

The following construction was given by MILNOR in his book [MS74].

1 **Definition: "Graßmann Manifold"**

For $n, k \in \mathbb{Z}_+$ we define the *Graßmann Manifold* as the set of all n-dimensional subspaces of $\mathbb{R}^{n+k}$. More precisely

$$\mathrm{Gr}_n(\mathbb{R}^{n+k}) := \mathrm{St}_n(\mathbb{R}^{n+k}) \big/ \sim, \tag{1.1}$$

where

$$\mathrm{St}_n(\mathbb{R}^{n+k}) := \left\{ (x_1, \dots, x_n) \in (\mathbb{R}^{n+k})^n \;\middle|\; (x_l) \text{ linearly independent} \right\} \tag{1.2}$$

is the *Stiefel manifold*, consisting of all Hamel bases of $\mathbb{R}^{n+k}$ and

$$(x_l) \sim (y_l) \Leftrightarrow \operatorname{span} x_l = \operatorname{span} y_l. \tag{1.3}$$

2 **Definition: "Canonical Vector Bundle"**

The *canonical vector bundle* over $\mathrm{Gr}_n(\mathbb{R}^{n+k})$ is

$$\gamma^n(\mathbb{R}^{n+k}) := (\left\{ (V, x) \in \mathrm{Gr}_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k} \;\middle|\; V \ni x \right\}, \pi_1, \mathrm{Gr}_n(\mathbb{R}^{n+k})), \tag{2.1}$$

so that the fibres are equal to their base points in the Graßmannian.

3 **Lemma: Representation of Vector Bundles**

For any vector bundle $p\colon E \to M$ of rank n over a compact space M (compact always meaning Hausdorff too) there is a continuous map $f\colon M \to \mathrm{Gr}_n(\mathbb{R}^{n+k})$ such that $f^*\gamma^n(\mathbb{R}^{n+k}) \simeq E$, provided k is chosen sufficiently big.

Proof (3) We start by constructing a map $\varphi\colon M \to \mathbb{R}^{n+k}$ which is injective on each fibre.

Choose a finite open cover $(U_1, \dots, U_r)$ of M such that all $E|_{U_l}$ are trivial. Then there are linear coordinate functions $\varphi_i\colon p^{-1}(U_l) \to \mathbb{R}^n$ which are injective on each fibre over U_l. Because M is compact and thus normal (i.e. T_4), we can extend the φ_l with a partition of unity to $\tilde{\varphi}_l$ over the whole of M. Then the combination

$$\varphi\colon E \ni e \mapsto (\tilde{\varphi}_1(e), \dots, \tilde{\varphi}_r(e)) \in (\mathbb{R}^n)^r \simeq \mathbb{R}^{rn} \tag{3.1}$$

is a continuous function that maps each fibre linearly and injective. This means $n + k := rn$ and we have to look at the Graßmannian $\mathrm{Gr}_n(\mathbb{R}^{rn})$. Now the desired function is

$$f\colon M \ni x \mapsto \varphi(p^{-1}(x)) \in \mathrm{Gr}_n(\mathbb{R}^{rn}) \tag{3.2}$$

which does map each point to an n-dimensional subspace of $\mathbb{R}^{rn}$. To see that indeed $E \simeq f^*\gamma^n(\mathbb{R}^{rn})$, we define a bundle map

$$g\colon E \ni e \mapsto (f(p(e)), \varphi(e)) \tag{3.3}$$

where

$$\varphi(e) \in \varphi(p^{-1}(p(e))) \overset{\text{def}}{=} f(p(e)) \Rightarrow g(e) = (f(p(e)), \varphi(e)) \in \gamma^n(\mathbb{R}^{rn}). \tag{3.4}$$

This bundle map is bijective between fibres. The following Lemma 4 now provides the isomorphism.

$$\begin{array}{ccc} E & \xrightarrow{g} & \gamma^n(\mathbb{R}^{rn}) \\ \downarrow p & \circlearrowleft & \downarrow \pi_1 \\ M & \xrightarrow{f} & \mathrm{Gr}_n(\mathbb{R}^{rn}) \end{array} \tag{3.5}$$

4 **Lemma: Bundle Maps and Pullback Bundles**

Given a bundle map g between $\pi_1\colon E_1 \to M_1$ and $\pi_2\colon E_2 \to M_2$ which is bijective between fibres, with base map f, such that the following diagram commutes, E_1 is isomorphic to f^*E_2.

$$\begin{array}{ccc} E_1 & \xrightarrow{g} & E_2 \\ \downarrow \pi_1 & \circlearrowleft & \downarrow \pi_2 \\ M_1 & \xrightarrow{f} & M_2 \end{array} \tag{4.1}$$

Proof (4) The pullback bundle is defined as

$$f^*E_2 = \left\{ (x, e) \in M_1 \times E_2 \;\middle|\; f(x) = \pi_2(e) \right\}. \tag{4.2}$$

We choose a map $\psi\colon E_1 \to f^*E_2$ by setting

$$\psi(e) := (\pi_1(e), g(e)). \tag{4.3}$$

Then because f is the base map corresponding to g (4.1) we know that

$$f(\pi_1(e)) = \pi_2(g(e)) \Leftrightarrow \psi(e) \in f^*E_2. \tag{4.4}$$

So because g is bijective between fibres and ψ respects the base point, ψ is everywhere bijective. Continuity is obvious because of local triviality.

5 **Definition: "Universality"**

A fibre bundle $E \to B$ is called *universal* for a category C of fibre bundles, if

1. For any fibre bundle $F \to M$ in C there is a continuous map $f\colon M \to B$ such that $f^*E \simeq F$.
2. Whenever $f^*E \simeq g^*E$ for two maps $f, g\colon M \to B$ also $f \simeq g$, i.e. f and g are homotopic.

2 Universal Principal Fibre Bundles

Now we want to look at the more general case of principal fibre bundles over paracompact spaces. Note however that the theory can also be developed for even more general spaces and bundles. (Compare [Mil56] and [Hus93].)

6 **Definition: "Infinite Join"**

Let G be any group. We start by construction the infinite join

$$\bigstar G := G \star G \star G \star \ldots := \left\{ (x,t) \in \bigoplus_{\mathbb{N}} (G \times [0,1]) \,\middle|\, \sum_{l\in\mathbb{N}} t_l = 1 \right\} \Big/ \sim \tag{6.1}$$

where

$$(x,t) \sim (x',t') \Leftrightarrow \forall l \in \mathbb{N}\colon t_l = t_{l'} \wedge (t_l \neq 0 \Rightarrow x_l = x_{l'}). \tag{6.2}$$

In particular for each point only finitely many t_l do not vanish. So $\bigstar G$ admits a filtration $G \subset G \star G \subset \ldots \subset \bigstar^n G \subset \ldots \subset \bigstar G$, where for $\bigstar^n G$ we require $t_l = 0$ for all $l > n$.

7 **Example:**

If G is the trivial group $\{0\}$, $\bigstar^{n+1} G$ is an n-simplex. For $G = \mathbb{Z}_2$ we have $\bigstar^{n+1} G \simeq S^n$.

8 **Definition: "Infinite Dimensional Principal Bundle"**

For a topological group G, $\bigstar G$ carries a natural topology. Furthermore we can define a continuous right G-action on $\bigstar G$ by

$$\bigstar G \times G \ni ((x,t), y) \mapsto (xy, t) \in \bigstar G. \tag{8.1}$$

Then

$$B_G := \bigstar G \big/ G \tag{8.2}$$

together with the natural quotient map $p_G\colon \bigstar G \to B_G$ lets us define the principal G-bundle $P_G := (\bigstar G, p_G, B_G)$.

9 **Lemma: Universal Bundles**

P_G is a universal principle bundle for the category of all principal G-bundles over paracompact spaces.

Proof (9) Let $p\colon P \to M$ be any principal G-bundle. As before, we construct a bundle map $g\colon P \to P_G$, just that we now have a locally finite countable cover. For principal bundles bijectivity between fibres is automatic and Lemma 4 (Bundle Maps and Pullback Bundles) gives the result.

If $f_1, f_2\colon M \to B_G$ are two continuous functions such that $f_0^* P_G \simeq f_1^* P_G \simeq P$. Then we need to construct a homotopy $f_s\colon M \to B_G$ from f_0 to f_1. First we let all even entries in P_G vanish. Let $I_n := [1 - (\frac{1}{2})^{n-1}, 1 - (\frac{1}{2})^n]$ and define the linear transformation $\alpha_n\colon I_n \ni t \mapsto 2^n t - 2^n + 2$.

Then we can define the following homotopy $P_G \times [0,1] \to P_G$ piecewise, for $s \in I_n$:

$$h_s^{od}(x,t) = (x',t') \tag{9.1}$$

$$(x'_l, t'_l) = \begin{cases} (x_l, t_l) & 1 \le l \le n \\ (x_{n+j}, \alpha_n(s) t_{n+j}) & l = n+2j-1 \\ (x_{n+j}, (1-\alpha_n(s)) t_{n+j}) & l = n+2j \end{cases} \tag{9.2}$$

This homotopy respects the action of G and thus induces a homotopy $g_s^{od}\colon B_G \to B_G$. In the same way we get a homotopy $h_s^{ev}\colon P_G \to P_G$ that lets all odd entries vanish. So now we can assume, that $f_0(M) \subset g_0^{od}(B_G)$ and $f_1(M) \subset g_0^{ev}(B_G)$. Then the final homotopy takes the form

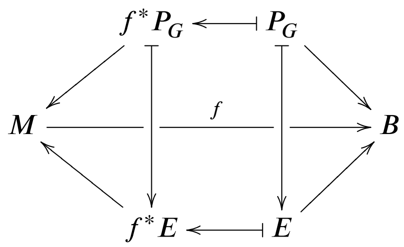
$$k\colon P \times [0,1] \ni (z,s) \mapsto (x,t) \in P_G \tag{9.3}$$

$$(x_l, t_l) = \begin{cases} (1-s)\tilde{f}_0(z) & 2 \nmid l \\ s\tilde{f}_1(z) & 2 \mid l \end{cases} \tag{9.4}$$

where we use the natural maps $\tilde{f}_{0/1}$ from P over $f_{0/1}^* P_G$ to P_G. Because $k(zy,s) = k(z,s)y$ for any $y \in G$, this induces a homotopy $M \to B_G$ between f_0 and f_1.

10 **Remark: Universal Fibre Bundles**

Now let $f\colon M \to B$ be the representation of a principal bundle $P \simeq f^* P_G$ and let E and E' be the associated bundles to P and P_G under the same G-action on F. Then $f^* E'$ and E are isomorphic. Thus E is a universal bundle for the category of fibre bundles with structure group G and fibre F.



(10.1)

3 Category Theoretic Perspective

11 **Definition: "Category of Principle Bundles"**

Let $\underline{\mathrm{H}}$ denote the category that has all paracompact topological spaces as objects, homotopy classes of continuous maps as morphisms and define $\underline{\mathrm{PB_G}}$ to be the category of isomorphism classes of principal G-bundles over paracompact spaces with the G-bundle maps as morphisms. Then the pullback is a contravariant functor $\underline{\mathrm{H}} \to \underline{\mathrm{PB_G}}$.

12 **Definition: "Representable Functors"**

Let C be a locally small category and $\underline{\mathrm{Set}}$ the category of all sets. If a contravariant functor $F\colon C \to \underline{\mathrm{Set}}$ is naturally isomorphic to the functor $\mathrm{Mor}(-,X)$ for some object $X \in C$, where

Mor(–, X) is the contravariant functor $C \to \underline{\text{Set}}$ that assigns to each object Y the morphisms $Y \to X$, we say F is *representable*. In this case X is called an *universal object*.

13 **Lemma: Representability Condition**

A function $F\colon C \to \underline{\text{Set}}$ as above is representable iff apart from the object $X \in C$ there is an element $u \in F(X)$, such that the map

$$\text{Mor}(Y, X) \ni f \mapsto F(f)(u) \in F(Y) \tag{13.1}$$

is bijective for all objects $Y \in C$.

This can be proven using the inverse of YONEDA*'s isomorphism*.

14 **Lemma: Universal Elements**

In our situation we have the pullback functor $\underline{\text{H}} \to \underline{\text{PB}_\text{G}}$ which can be made into a functor $F\colon \underline{\text{H}} \to \underline{\text{Set}}$, by assigning to every paracompact space the set of principal G-bundles over it and to each continuous function, the element-wise pullback. Now what we showed in Lemma 9 (Universal Bundles) is, that F is representable with universal element B_G.

Proof (14) According to Lemma 13 (Representability Condition) we can take the bundle $u := \star G \to B_G \in \underline{\text{PB}_\text{G}}(B_G)$ and show that the map

$$\text{Mor}(Y, B_G) \ni f \mapsto f^*\star G \in \underline{\text{PB}_\text{G}}(Y) \tag{14.1}$$

is bijective for all paracompact spaces Y. But this is exactly our Definition 5 (Universality)

References

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