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Diplomarbeit

Ein Vergleich zwischen dem Melrose b-Kalkül und dem Ballmann-Brüning-Carron-Kalkül für Dirac-Systeme

A Comparison between the Melrose b-Calculus and the Ballmann-Brüning-Carron-Calculus of Dirac Systems

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1. Introduction

The classic Atiyah-Patodi-Singer index theorem established an index formula for manifolds M with cylindrical ends under certain non-local boundary conditions [APS74]. Later, by using a singular, so called b-metric

$$\frac{(\mathrm{d}x)^2}{x^2} + g_{\partial M}, \tag{1-1}$$

where x is a boundary defining function, Melrose developed a different approach for understanding such index formulas (see [Mel93, Chapter 2] and [Mel96, Section 2.2]). He also considers more general settings in [Mel93, Section 8.1 and 8.2] as well as in [Mel96, Section 2.5] than what we are currently interested in.

When we impose the additional condition on M, that its boundary $N \coloneqq \partial M$ is a fibre bundle (denoted by $(N, \phi, B; \mathcal{F})$), adapted singular metrics arise, which might be analysed in a similar fashion to the b-metric. Among them is for instance the *ϕ-metric*

$${}^{\phi}g = \frac{(\mathrm{d}x)^2}{x^4} + \frac{\phi^* g_B}{x^2} + g_V, \tag{1-2}$$

which was first defined in [MM98, Sections 1 and 8]. Then Vaillant made extensive use of it as a means to prove an index theorem for the related d-metric (the ϕ-metric multiplied by x^2) in [Vai01]. For that purpose he constructed a pseudo-differential calculus, which is a highly involved process.

Our first major point of reference is the paper by Leichtnam, Mazzeo and Piazza [LMP06]. There the authors prove an index theorem for the Dirac operator over manifolds with a ϕ-metric by continuously transforming the ϕ-metric into a b-metric. While they acknowledge their result might have been obtained by methods similar to those of Vaillant, they arrive at their result in only a dozen pages. The index formula they find is [LMP06, Equation (1.4)]

$$\operatorname{ind} D^+ = \int_M \hat{A}(M, {}^{\phi}g) \wedge \operatorname{ch} E - \tfrac{1}{2} \int_B \hat{A}(B, g_B) \wedge \tilde{\eta} \tag{1-3}$$

for a coefficient bundle E. What we notice here, is that this index formula has the form

$$\operatorname{ind} D^+ = \int_M \hat{A}(M, {}^{\phi}g) \wedge \operatorname{ch} E + \sum_C \operatorname{Corr}(C) \tag{1-4}$$

with $\operatorname{Corr}(C)$ being correction terms for the ends C of M. This is exactly the form of index theorems [BBC12, Theorem 1.17] which appear in the work of Ballmann, Brüning and Carron based on the notion of *non-parabolicity at infinity* and their previous work on manifolds with straight ends [BBC08].

Our goal is to prove that these similarities are not coincidental and to derive the ϕ-index formula as an application of the methods of [BBC12]. We do this by transforming the situation in [LMP06] into one that is compatible with [BBC12]. Then we give definitions and properties of the important objects involved in the latter. After that we can prove that the resulting Dirac System is of Fredholm type, which allows us to employ the results in [BBC12] to calculate the expected index formula.

When considering the ϕ-metric in a more algebraic sense, especially when developing the ϕ-pseudo-differential-calculus, it is natural to consider the ϕ-metric as a regular bundle metric on the ϕ-tangent bundle of M with respect to x. See [Gra17] for an introduction using this method including complete proofs. [Vai01, Section 1] also discusses this topic.

In contrast, we start section 2 by introducing the product ϕ-metric in a simpler, more direct way. This metric makes M non-compact, effectively pushing the boundary to infinity. To account for this behaviour we go on to transform the boundary defining function x into a distance function $f := \frac{1}{x}$ with associated unit vector field $T := \operatorname{grad} f$. Thus we can describe the end as a product $M_\infty := I \times N \ni (t, p)$: it is straight as defined in [BBC12, Definition 1.8]. Using that product structure, we can calculate parallel transport and the t-dependence of the Christoffel symbols as well as of the curvature.

In section 3, after giving the definition of a Dirac bundle, we construct the restricted Dirac bundle. Most importantly, it allows us to write the Dirac operator in the form

$$D = \mathrm{Cl}(T)(\nabla^E_T - D^t + \frac{\kappa^t}{2}). \tag{1-5}$$

By employing parallel continuation in T-direction of sections of the Dirac bundle E, just as described in [BBC12, § 3.3], we find a product structure on E (see section 4). This lets us write the above decomposition in the following form, very similar to the one in [APS74]:

$$D = \mathrm{Cl}(T)(\frac{d}{dt} + \frac{\kappa}{2} + A^t), \tag{1-6}$$

where A^t corresponds to $-D^t$ and leads to the construction of a Dirac system.

After this general setup we restrict our considerations to even-dimensional spin manifolds, as [LMP06] do, for all of section 5. In addition we only consider the spinor bundle $E := \mathcal{S}(M)$. Other Dirac bundles can locally be written as $\mathcal{S}(M) \otimes W$ for some bundle W that carries a flat connection and is not affected by the Clifford multiplication. In such a local case not even a spin-structure is needed, so that our mostly local calculations can likely be generalized to any (super-symmetric) Dirac bundle.

In subsection 5.1 firstly we define the spinor bundle together with its natural grading $\mathcal{S}^+(M) \oplus \mathcal{S}^-(M)$. There, most importantly, we can give an explicit formula for the connection induced by the Levi-Civita connection. As a side-fact we find that when restricting the spin structure to N and associating a spinor bundle to it via the isomorphism $\mathrm{Cl}_{m-1} \to \mathrm{Cl}^+_m$, the resulting Dirac bundle is exactly the restricted Dirac bundle from section 3.

In [Brü09, Section 1] Brüning analyses Dirac operators over fibre bundles like D^t over N^t. We employ this technique in subsection 5.2 to decompose A^t into vertical and horizontal parts roughly as follows:

$$-A^t = A_V + \tfrac{1}{t}A_H - \tfrac{1}{t^2}\beta. \tag{1-7}$$

Only here the statement in [LMP06] becomes apparent, that the *boundary operator* induces a family of vertical operators $D^\partial_y \triangleq A^\infty_V$, because the horizontal derivatives vanish for $t \to \infty$. Just as in [LMP06, Assumption 1.1] we need to presuppose that those A^∞_V are invertible with a common lower bound.

Still following [Brü09] we consider the anti-commutator A_{HV} of A_H and A_V (up to perturbation by endomorphisms) and confirm that it is a first order vertical operators. This allows us to estimate it by A_V. Using this estimate we can prove the preconditions of [BBC12, Proposition 4.40], which yields the non-parabolicity of D and—in our case—the even stronger fact that D is of *Fredholm type*. Similar arguments give us that all A^t are invertible and that [BBC12, Proposition 4.46] holds. That is the reason why we can apply the results from [BBC12, § 5].

Finally we go on to calculate the index formula in subsection 5.3 by using the results of [BBC12, § 5]. First we get

$$\begin{aligned}\operatorname{ind} D^+_{\mathrm{ext}} = \int_{M^\tau} \hat{A}(M, {}^\phi g) + \int_{N^\tau} T\hat{A}(\nabla, \tilde{\nabla}^\tau) + \tfrac{1}{2}(\eta(A^{\tau,+}) + \dim\ker A^{\tau,+})\\ + \dim H^+_{[-\lambda,0)} + \operatorname{ind} D^+_{U^\tau,<-\lambda,\mathrm{ext}}\end{aligned} \tag{1-8}$$

using the transgression to an APS-like situation near N^τ. The last two terms vanish immediately. Then taking the limit for $\tau \to \infty$ we get the Bismuth-Cheeger $\tilde{\eta}$-form and find that the transgression term vanishes. The resulting index formula is

$$\operatorname{ind} D^+_{\mathrm{ext}} = \int_M \hat{A}(M, {}^\phi g) - \tfrac{1}{2}\int_B \hat{A}(B, g_B) \wedge \tilde{\eta} \tag{1-9}$$

which is exactly the initially cited index formula from [LMP06] for the spinor bundle.

Notation We will consider only smooth objects, unless stated otherwise. Also there should be no confusion from the fact that instead of a capital Φ like in [MM98] and [LMP06] we follow the notation of [Vai01] and write a lower case ϕ matching the choice for the related b-, c- and d-metrics.

List of Symbols

$\{A, B\}$	Anti-Commutator of A and B; $A \circ B + B \circ A$	32
$(\sigma_1, \sigma_2)_t$	t-scalar product for sections of E^{t_0}	21
Ad_m	Adjoint Representation of Spin_m on SO_m	23
A^t_H	Symmetrized Horizontal Part of A^t	29
A'	Commutator of $\frac{d}{dt}$ and A^t	30
A^t	Restricted Dirac Operator in the Dirac System; $-\mathrm{P}\, D^t$ ꟼ	22
A^t_V	Symmetrized Vertical Part of A^t	29
B	Base of the Fibre Bundle N	10
Cl	Clifford Multiplication	16
Cl^t	Restricted Clifford Multiplication	18
Cl_k	Clifford Algebra over $\mathbb{R}^k$	23
D	Dirac Operator	16
(e_i)	Local Orthonormal and ϕ-Subordinate Frame of N^{t_0}	13
$(\bar{e}_i)$	Parallel Continuation of (T, e_i) in T-direction	13
f	Distance Function	11
$\mathcal{F}$	Fibre of the Fibre Bundle N	10
g_B	Riemannian Metric on B	10
g_∞	ϕ-Metric on M_∞	11
g^t	g_∞ restricted to N^t	12
g_V	Bundle Metric on $T^V N$	10
h	Metric on $\mathcal{S}(M)$	23
h^E	Hermitian Bundle Metric on E	16
I	Interval of the Dirac System	11
id_E	Identity Map on E	16
$\imath$	Imaginary Unit	17
m	Dimension of M	10
M_∞	Product Model of the Ends of M	11
N	Compact Boundary of M	10
N^t	Slice of M_∞; $\{t\} \times N$	12
P^E	Parallel Transport in T-direction in a Bundle E	20
ꟼ^E	Inverse of P^E	20
R^l_{ijk}	Curvature Coefficients	15
$\mathcal{S}_m$	Spinor Module	23
$\mathrm{SO}(M)$	SO_m-Principal Bundle of M	23
$\mathrm{Spin}(M)$	Spin_m-Principal Bundle of M	23
T	Gradient Vector Field of f	12
$T^H N$	Horizontal Distribution; Orthogonal Complement of $T^V N$	10

2. Pushing the Boundary to Infinity

Let M be an m-dimensional manifold with compact boundary $N = \partial M$, such that the boundary is a fibre bundle $(N, \phi, B; \mathcal{F})$. Assume there is a metric g_N on N (not induced by a metric on M) and that ϕ is a Riemannian submersion. This means that g_N restricted to the orthogonal complement $T^H N$ of $T^V N \coloneqq \ker \mathrm{d}\phi$ is equal to $\phi^* g_B$ for a metric g_B on B uniquely determined by g_N.

Using the projection onto the summands in the orthogonal decomposition $T^H N \oplus T^V N$ of TN we can then write

$$g_N = \phi^* g_B + g_V. \tag{2-1}$$

Furthermore let there be a fixed collar $\bar{U} \subset M$ around N with collar isomorphism $\Psi\colon [0, C] \times N \to \bar{U}$ and associated boundary defining function x. This lets us write TM over $\bar{U}$ as $T[0, C] \oplus T^H N \oplus T^V N$. Now assume that there is a product ϕ-metric ${}^\phi g$ on M, i.e. ${}^\phi g$ takes the following form over the interior $U \coloneqq \mathring{\bar{U}} = \Psi((0, C) \times N)$:

$$\Psi^*({}^\phi g) = \frac{(\mathrm{d}x)^2}{x^4} + \frac{\phi^* g_B}{x^2} + g_V. \tag{2-2}$$

By ∇ denote the ϕ-Levi-Civita connection for this metric, which is the usual Levi-Civita connection on the interior $\mathring{M}$.

The chosen metric, when regarded as a standard Riemannian metric, is singular on the boundary. For that reason previous work employed complicated, custom ϕ-pseudo-differential-calculi to analyse differential operators and their indices. Our goal is to use methods that apply to standard Riemannian manifolds. In particular we will use the methods of Dirac systems given in [BBC08] and [BBC12].

If we restrict ${}^\phi g$ to the interior of our manifold, there are no more singularities, but the result $(\mathring{M}, {}^\phi g)$ is non-compact. Rescaling the collar to account for this fact is what we call *Pushing the Boundary to Infinity*.

In this section we will show that M has straight ends as defined in [BBC12, Definition 1.8] by rescaling Ψ to yield the distance function $\frac{1}{x}$. After that we use the rescaled product model M_∞ of the ends of M to calculate parallel transport, Levi-Civita connection and curvature for the product ϕ-metric ${}^\phi g$.

1. Definition: "Distance Function"

A function $f\colon M \to \mathbb{R}$ on a Riemannian manifold (M, g) is called a *distance function*, if grad f is a unit vector field.

2. Lemma:

If f is a distance function on (M, g) and $T \coloneqq \operatorname{grad} f$ is its gradient vector field, then the integral curves of T are geodesics.

Proof Let $X \in \mathcal{V}(M)$ be any vector field. Note that according to the definition $g(T, X) = \mathrm{d}f(X) = X(f)$ and likewise $T(f) = g(T, T) = 1$. Now according to the Koszul formula

$$\begin{aligned} g(\nabla_T T, X) &= T(g(T, X)) - \tfrac{1}{2} X(g(T, T)) - g(T, [T, X]) \\ &= T(\mathrm{d}f(X)) - \mathrm{d}f([T, X]) \\ &= T(X(f)) - T(X(f)) + X(T(f)) = X(g(T, T)) = 0. \end{aligned} \tag{2.1}$$

So the integral curves γ of T satisfy $\nabla_{\dot\gamma}\dot\gamma = \nabla_T T = 0$, i.e. are geodesics. ☑

In order to apply methods from [BBC12] and related work, we need a distance function f, while the ϕ-metric is defined in terms of a boundary defining function x.

3. Lemma: From Boundary Defining to Distance Functions

The function defined by

$$f \colon U \ni p \mapsto \frac{1}{x(p)} \in (\frac{1}{C}, \infty) =: I \tag{3.1}$$

is a distance function on $(U, {}^{\phi}g)$.

Proof For convenience we work on $(0, C) \times N$ instead of U and keep Ψ implicit. Then $f(x, q) = \frac{1}{x}$ and so

$$\mathrm{d}f_{(x,q)} = -\frac{1}{x^2}\mathrm{d}x_{(x,q)}. \tag{3.2}$$

Then we can write the metric as

$${}^{\phi}g = (\mathrm{d}f)^2 + f^2 \phi^* g_B + g_V. \tag{3.3}$$

Furthermore f is constant on any $\{x\} \times N$, which gives us

$$\mathrm{d}f|_{T(\{x\}\times N)} = 0 \tag{3.4}$$

$$\Leftrightarrow \operatorname{grad} f \perp T(\{x\} \times N) \tag{3.5}$$

$$\Rightarrow \mathrm{d}f(\operatorname{grad} f) \overset{\text{def}}{=} {}^{\phi}g(\operatorname{grad} f, \operatorname{grad} f) = (\mathrm{d}f(\operatorname{grad} f))^2. \tag{3.6}$$

Now $\mathrm{d}f \neq 0$, thus $\operatorname{grad} f$ cannot vanish and ${}^{\phi}g(\operatorname{grad} f, \operatorname{grad} f) = 1$. That means f is a distance function. ☑

4. Definition: “Product Model”

The gradient flow of f is described by the adapted isomorphism

$$F \colon M_\infty := I \times N \ni (t, p) \mapsto \Psi(\frac{1}{t}, p) \in U \tag{4.1}$$

with which we can pull back the product ϕ-metric as

$$g_\infty := F^*({}^{\phi}g) = (\mathrm{d}t)^2 + t^2 \phi^* g_B + g_V. \tag{4.2}$$

This means F is an isometry between the Riemannian Manifolds $(U, {}^{\phi}g)$ and (M_∞, g_∞). Now in terms of [BBC12, Definition 1.8] we can say that M *has straight ends*.

For $t \in I$ we define the level surface

$$N^t := \{t\} \times N \subset M_\infty \tag{4.3}$$

as fibre bundle $(N^t, \phi^t := (\mathrm{id}_I, \phi), B^t := \{t\} \times B; \mathcal{F})$ with the induced Riemannian metric

$$g^t := g_\infty\big|_{TN^t}. \tag{4.4}$$

Furthermore we will write

$$T := \operatorname{grad} f \circ F \tag{4.5}$$

as in [BBC12, § 3.2] for the unit vector field in t-direction.

5. Lemma: Parallel Transport

Let X and Y be vector fields on M_∞ of the form $X = (0, \tilde{X})$ and $Y = (0, \tilde{Y})$ for $\tilde{X}, \tilde{Y} \in \mathcal{V}(N)$ — independent of t — with $\tilde{X}$ horizontal and $\tilde{Y}$ vertical. Then the Levi-Civita connection in T-direction takes the form

$$\nabla_T X = \tfrac{1}{t} X, \qquad \nabla_T Y = 0 \tag{5.1}$$

and the parallel transport along the geodesics $\gamma_p \colon I \ni t \mapsto (t, p) \in M_\infty$ from t_0 to t_1 is

$$P^{t_0,t_1}_{\gamma_p}(X) = \tfrac{t_0}{t_1} X, \qquad P^{t_0,t_1}_{\gamma_p}(Y) = Y. \tag{5.2}$$

Proof First let $X = (0, \tilde{X})$ and $Y = (0, \tilde{Y})$ be arbitrary. Because of this form X (and Y respectively) and T have commuting flows. So we know that $[T, X] = [T, Y] = 0$ as well as $X, Y \perp T$ and the Koszul formula shows

$$\begin{aligned} g_\infty(\nabla_T X, T) &= \tfrac{1}{2}(T(g_\infty(X,T)) + X(g_\infty(T,T)) - T(g_\infty(T,X)) \\ &\quad - g_\infty(T,[X,T]) + g_\infty(X,[T,T]) + g_\infty(T,[T,X])) \\ &= \tfrac{1}{2} X(\underbrace{g_\infty(T,T)}_{\text{constant}}) + g_\infty(T, \underbrace{[T,X]}_{=0}) = 0 \end{aligned} \tag{5.3}$$

$$\begin{aligned} g_\infty(\nabla_T X, Y) &= \tfrac{1}{2}(T(g_\infty(X,Y)) + X(g_\infty(Y,T)) - Y(g_\infty(T,X)) \\ &\quad - g_\infty(T,[X,Y]) + g_\infty(X,[Y,T]) + g_\infty(Y,[T,X])) \\ &= \tfrac{1}{2} \frac{d}{dt} g^t(X,Y) = t g_H(X,Y), \end{aligned} \tag{5.4}$$

where we write $g_H := \phi^* g_B$.

This means vertical vector fields X are parallel in T-direction if they are a constant family of vector fields on N. For X horizontal we get $\nabla_T X \parallel X$ and thus

$$t^2 g_H(\nabla_T X, X) = g_\infty(\nabla_T X, X) \overset{(5.4)}{=} t g_H(X,X) \tag{5.5}$$

$$\Rightarrow t^2 \nabla_T X = tX \Rightarrow \nabla_T X = \tfrac{1}{t} X. \tag{5.6}$$

Now for horizontal vector fields varying in t, i.e. $f \cdot X$ with $f\colon I \to \mathbb{R}$, this implies

$$\nabla_T f(t)X = f'(t)X + f(t)\nabla_T X = f'(t)X + \tfrac{1}{t} f(t)X \tag{5.7}$$

$$\nabla_T f(t)X = 0 \Leftrightarrow f'(t) = -\tfrac{1}{t} f(t) \Leftrightarrow \exists C\colon f(t) = \tfrac{1}{t} C \tag{5.8}$$

$$\Rightarrow P^{t_0,t_1}(X) = \tfrac{t_0}{t_1} X \tag{5.9}$$

☑

6. Definition: "ϕ-Subordinate Local Frame"

For our fibre bundle $(N, \phi, B; \mathcal{F})$ with horizontal distribution $T^H N$ we say that a local frame $(e_1, \dots, e_{\dim N})$ of TN over some open neighbourhood $U \subset N$ is *ϕ-subordinate*, if $e_1, \dots, e_{\dim B}$ are horizontal of the form $\phi^* e_i^B, e_i^B \in \mathcal{V}(B)$ and $e_{1+\dim B}, \dots, e_{\dim N}$ are vertical. Because ϕ is a Riemannian submersion, by choosing (e_i^B) orthonormal, we can construct local orthonormal ϕ-subordinate frames.

We will furthermore call the indices $i \in \{1, \dots, \dim B\}$ *horizontal*, $j \in \{\dim B + 1, \dots, \dim N\}$ *vertical* and write e_i^V as well as e_j^H.

7. Lemma: Levi-Civita Connection

Choose any $t_0 \in \mathbb{R}_+$ and take a local orthonormal ϕ-subordinate frame $(e_2, \dots, e_{\dim M})$ of TN^{t_0}. Then we have

$$[e_i, e_j] \in \Gamma(TN) \quad \Rightarrow \quad g_\infty([e_i, e_j], T) = 0 \tag{7.1}$$

$$[e_i^V, e_j] \in \Gamma(T^V N) \quad \Rightarrow \quad g_\infty([e_i^V, e_j], e_k^H) = 0. \tag{7.2}$$

Now we extend $(e_i)_i = (T, e_2^H, \dots, e_{\dim B+1}^H, e_{\dim B+2}^V, \dots, e_{\dim M}^V)$ via parallel transport along the geodesics in T direction

$$\bar{e}_i^H := P_T^{t_0,t} e_i^H = \tfrac{t_0}{t} e_i^H, \quad \bar{e}_i^V := P_T^{t_0,t} e_i^V = e_i^V \tag{7.3}$$

to a local orthonormal frame $(\bar{e}_1, \dots, \bar{e}_{\dim M})$ of M. In this frame the Christoffel symbols are $\Gamma_{i,j}^k(t) := g_\infty(\nabla_{\bar{e}_i} \bar{e}_j, \bar{e}_k)\Big|_{N^t}$. By using the facts that

$$g^t(e_i^H, X) = t^2 \phi^* g_B(\mathrm{d}\phi(e_i^H), \mathrm{d}\phi(X)) = \tfrac{t^2}{t_0^2} g^{t_0}(e_i^H, X) \tag{7.4}$$

$$g^t(e_i^V, X) = g_V(e_i^V, X) = g^{t_0}(e_i^V, X) \tag{7.5}$$

$$g^t(T, X) = \mathrm{d}f(X) = g^{t_0}(T, X) \tag{7.6}$$

$$[\bar{e}_i^H, T] = [\tfrac{t_0}{t} e_i^H, T] = \tfrac{t_0}{t^2} e_i^H = \tfrac{1}{t} \bar{e}_i^H \tag{7.7}$$

we can calculate with the Koszul formula

$$\begin{aligned} 2g^t(\nabla_{\bar e_i^H} \bar e_j^H, \bar e_k^H) &= -\tfrac{t_0}{t} g^{t_0}(e_i^H,[e_j^H,e_k^H]) + \tfrac{t_0}{t} g^{t_0}(e_j^H,[e_k^H,e_i^H]) + \tfrac{t_0}{t} g^{t_0}(e_k^H,[e_i^H,e_j^H]) \\ &= 2\tfrac{t_0}{t} g^{t_0}(\nabla_{e_i^H} e_j^H, e_k^H) \end{aligned} \tag{7.8}$$

$$\begin{aligned} 2g^t(\nabla_{\bar e_i^H}\bar e_j^H, \bar e_k^V) &= \cancel{-g^{t_0}(e_i^H,[e_j^H,e_k^V])} + \cancel{g^{t_0}(e_j^H,[e_k^V,e_i^H])} + \tfrac{t_0^2}{t^2} g^{t_0}(e_k^V,[e_i^H,e_j^H]) \\ &= 2\tfrac{t_0^2}{t^2} g^{t_0}(\nabla_{e_i^H} e_j^H, e_k^V) \end{aligned} \tag{7.9}$$

$$\begin{aligned} 2g^t(\nabla_{\bar e_i^H}\bar e_j^V,\bar e_k^V) &= \cancel{-\tfrac{t}{t_0} g^{t_0}(e_i^H,[e_j^V,e_k^V])} + \tfrac{t_0}{t} g^{t_0}(e_j^V,[e_k^V,e_i^H]) + \tfrac{t_0}{t} g^{t_0}(e_k^V,[e_i^H,e_j^V]) \\ &= 2\tfrac{t_0}{t} g^{t_0}(\nabla_{e_i^H} e_j^V, e_k^V) \end{aligned} \tag{7.10}$$

$$\begin{aligned} 2g^t(\nabla_{\bar e_i^V}\bar e_j^H,\bar e_k^H) &= -\tfrac{t_0^2}{t^2} g^{t_0}(e_i^V,[e_j^H,e_k^H]) + \cancel{g^{t_0}(e_j^H,[e_k^H,e_i^V])} + \cancel{g^{t_0}(e_k^H,[e_i^V,e_j^H])} \\ &= 2\tfrac{t_0^2}{t^2} g^{t_0}(\nabla_{e_i^V} e_j^H, e_k^H) \end{aligned} \tag{7.11}$$

$$\begin{aligned} 2g^t(\nabla_{\bar e_i^V}\bar e_j^H,\bar e_k^V) &= -\tfrac{t_0}{t} g^{t_0}(e_i^V,[e_j^H,e_k^V]) + \cancel{\tfrac{t}{t_0} g^{t_0}(e_j^H,[e_k^V,e_i^V])} + \tfrac{t_0}{t} g^{t_0}(e_k^V,[e_i^V,e_j^H]) \\ &= 2\tfrac{t_0}{t} g^{t_0}(\nabla_{e_i^V} e_j^H, e_k^V) \end{aligned} \tag{7.12}$$

$$\begin{aligned} 2g^t(\nabla_{\bar e_i^V}\bar e_j^V,\bar e_k^V) &= -g^{t_0}(e_i^V,[e_j^V,e_k^V]) + g^{t_0}(e_j^V,[e_k^V,e_i^V]) + g^{t_0}(e_k^V,[e_i^V,e_j^V]) \\ &= 2g^{t_0}(\nabla_{e_i^V} e_j^V, e_k^V) \end{aligned} \tag{7.13}$$

$$\begin{aligned} 2g^t(\nabla_{\bar e_i^H}\bar e_j^H, T) &= -g^t(\bar e_i^H,[\bar e_j^H,T]) + g^t(\bar e_j^H,[T,\bar e_i^H]) + g^t(T,[\bar e_i^H,\bar e_j^H]) \\ &= -\tfrac{1}{t}\delta_{ij} - \tfrac{1}{t}\delta_{ij} + 0 = 2\tfrac{t_0}{t} g^{t_0}(\nabla_{e_i^H} e_j^H, T) \end{aligned} \tag{7.14}$$

$$g_\infty(\nabla_{\bar e_i^H}\bar e_j^V, T) = g_\infty(\nabla_{\bar e_i^V}\bar e_j^H, T) = g_\infty(\nabla_{\bar e_i^V}\bar e_j^V, T) = 0. \tag{7.15}$$

Of course $\nabla_T \bar e_i = 0$ by definition of parallel transport.

8. Corollary: Symmetries of the Christoffel Symbols

Because ∇ is a metric connection, we immediately have

$$\Gamma^k_{i,j}(t) = -\Gamma^j_{i,k}(t) \quad \text{and} \quad \Gamma^j_{i,j}(t) = 0. \tag{8.1}$$

Furthermore let i, j be vertical and k be horizontal, then we conclude from (7.12) that

$$\Gamma^j_{i,k}(t) = \Gamma^i_{j,k}(t) \quad \text{and} \quad \Gamma^k_{i,j}(t) = \Gamma^k_{j,i}(t). \tag{8.2}$$

9. Lemma: Curvature

Lemma 7 shows, that the Christoffel symbols are polynomials in $\frac{1}{t}$. Thus so are the curvature coefficients R^l_{ijk} defined by $R(\bar{e}_i, \bar{e}_j)\bar{e}_k = \sum_l R^l_{ijk}\bar{e}_l$ where $R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$.

Proof First calculate, because ∇ is torsion free

$$[\bar{e}_i, \bar{e}_j] = \nabla_{\bar{e}_i}\bar{e}_j - \nabla_{\bar{e}_j}\bar{e}_i = \sum_n (\Gamma^n_{ij} - \Gamma^n_{ji})\bar{e}_n, \tag{9.1}$$

then the curvature is

$$\begin{aligned} R(\bar{e}_i, \bar{e}_j)\bar{e}_k &= \nabla_{\bar{e}_i}\nabla_{\bar{e}_j}\bar{e}_k - \nabla_{\bar{e}_j}\nabla_{\bar{e}_i}\bar{e}_k - \nabla_{[\bar{e}_i,\bar{e}_j]}\bar{e}_k \\ &= \nabla_{\bar{e}_i}\sum_l \Gamma^l_{jk}\bar{e}_l - \nabla_{\bar{e}_j}\sum_l \Gamma^l_{ik}\bar{e}_l - \sum_n (\Gamma^n_{ij} - \Gamma^n_{ji})\nabla_{\bar{e}_n}\bar{e}_k \\ &= \sum_l ((\bar{e}_i\Gamma^l_{jk})\bar{e}_l + \Gamma^l_{jk}\nabla_{\bar{e}_i}\bar{e}_l - (\bar{e}_j\Gamma^l_{ik})\bar{e}_l - \Gamma^l_{ik}\nabla_{\bar{e}_j}\bar{e}_l) - \sum_{n,l}(\Gamma^n_{ij} - \Gamma^n_{ji})\Gamma^l_{nk}\bar{e}_l \\ &= \sum_l (\bar{e}_i\Gamma^l_{jk} - \bar{e}_j\Gamma^l_{ik} + \sum_n (\Gamma^n_{jk}\Gamma^l_{in} - \Gamma^n_{ik}\Gamma^l_{jn} - (\Gamma^n_{ij} - \Gamma^n_{ji})\Gamma^l_{nk}))\bar{e}_l, \end{aligned} \tag{9.2}$$

where in the last step we exchanged the index l in $\Gamma^l_{jk}\nabla_{\bar{e}_i}\bar{e}_l$ and $\Gamma^l_{ik}\nabla_{\bar{e}_j}\bar{e}_l$ with n in order to use l for expansion into Christoffel symbols in all instances. ■

10. Corollary: Curvature Bounds

As t goes to ∞ on the ends, the above lemma shows that the curvature of ∇ is uniformly bounded on M. Similarly there is a uniform bound on the curvature of the Levi-Civita-Connection on the spinor bundle. So we have shown that [BBC12, Equations (1.1)] are full-filled when E is the spinor bundle.

According to [Vai01, Proposition 1.5] the ϕ-Levi-Civita connection is a connection in the usual sense, i.e. maps into $\Gamma(T^*M \otimes {}^{\phi}TM)$, for exact ϕ-metrics. That is why the boundedness of the curvature does not come unexpected.

3. Dirac Bundles

We begin this section by defining the general term of a *Dirac bundle* and its associated *Dirac operator*, followed by stating their basic properties. After that we can look at properties more specific to our setting of M_∞, for example the *restricted Dirac bundle*. This corresponds to [BBC12, § 2.1]. We will later see in subsection 5.1 how the spinor bundles, over M_∞ and the N^t, naturally display such a structure.

11. Definition: "Dirac Bundle, Dirac Operator"

Given a Riemannian manifold (M, g), a smooth $\mathbb{C}$-vector bundle $(E, \pi, M, \mathbb{C}^\eta)$ with Hermitian metric h^E and metric connection ∇^E is called a *Dirac bundle* if it carries a Clifford multiplication $\mathrm{Cl} \in \Omega^1(M, \mathrm{End}(E))$. That means, the following Clifford relations have to be satisfied:

$$\mathrm{Cl}(X) \circ \mathrm{Cl}(X) = -|X|^2_{TM}\,\mathrm{id}_E, \tag{11.1}$$

$$|\mathrm{Cl}(X)(\sigma)|_E = |X|_{TM} \cdot |\sigma|_E \tag{11.2}$$

and

$$\nabla^E_X(\mathrm{Cl}(Y)\sigma) = \mathrm{Cl}(\nabla_X Y)\sigma + \mathrm{Cl}(Y)(\nabla^E_X\sigma) \tag{11.3}$$

for all $X, Y \in \mathcal{V}(M)$ and $\sigma \in \Gamma(M, E)$. The associated *Dirac operator* is given by

$$D\sigma := \sum_{i=1}^m \mathrm{Cl}(e_i)(\nabla^E_{e_i}\sigma) \tag{11.4}$$

when $(e_i)_{i=1,\dots,m}$ is a local orthonormal frame of TM.

Proof D is well-defined independent of the frame (e_i), because if (f_j) is another local orthonormal frame on the same coordinate patch, then

$$\begin{aligned} D\sigma &= \sum_{i=1}^m \mathrm{Cl}(e_i)(\nabla^E_{\sum_{j=1}^n g(e_i,f_j)f_j}\sigma) = \sum_{i=1}^m\sum_{j=1}^m g(e_i,f_j)\,\mathrm{Cl}(e_i)\nabla^E_{f_j}\sigma \\ &= \sum_{j=1}^m \mathrm{Cl}(\sum_{i=1}^n g(e_i,f_j)e_i)\nabla^E_{f_j}\sigma = \sum_{j=1}^m \mathrm{Cl}(f_j)\nabla^E_{f_j}\sigma. \end{aligned} \tag{11.5}$$

∎

Often the properties of the Clifford multiplication are used in the following more general forms.

12. Remark: Polarization

The simplest polarization arises from (11.1) and yields

$$\mathrm{Cl}(X) \circ \mathrm{Cl}(Y) + \mathrm{Cl}(Y) \circ \mathrm{Cl}(X) = -2g(X,Y)\,\mathrm{id}_E, \tag{12.1}$$

which contains the fundamental notion that the Clifford multiplications by orthogonal vectors anti-commute.

Applying polarization to (11.2) in X gives

$$|\mathrm{Cl}(X+Y)\sigma| = |X+Y||\sigma| \tag{12.2}$$

$$\begin{aligned}\Rightarrow&|\mathrm{Cl}(X)\sigma|^2 + 2\,\mathrm{Re}\,h^E(\mathrm{Cl}(X)\sigma, \mathrm{Cl}(Y)\sigma) + |\mathrm{Cl}(Y)\sigma|^2\\ &= (|X|^2 + 2g(X,Y) + |Y|^2)|\sigma|^2\end{aligned} \tag{12.3}$$

$$\Rightarrow \mathrm{Re}\,h^E(\mathrm{Cl}(X)\sigma, \mathrm{Cl}(Y)\sigma) = g(X,Y)|\sigma|^2 \tag{12.4}$$

and in σ gives

$$|X|_{TM}|\sigma_1 - \mathrm{Cl}(X)\sigma_2| = |\mathrm{Cl}(X)(\sigma_1 - \mathrm{Cl}(X)\sigma_2)| \overset{(11.1)}{=} |\mathrm{Cl}(X)\sigma_1 + |X|^2_{TM}\sigma_2| \tag{12.5}$$

$$\begin{aligned}\Rightarrow&|X|^2_{TM}(|\sigma_1|^2 - 2\,\mathrm{Re}\,h^E(\sigma_1, \mathrm{Cl}(X)\sigma_2) + |\mathrm{Cl}(X)\sigma_2|^2)\\ &= |\mathrm{Cl}(X)\sigma_1|^2 + 2|X|^2_{TM}\,\mathrm{Re}\,h^E(\mathrm{Cl}(X)\sigma_1, \sigma_2) + |X|^4_{TM}|\sigma_2|^2\end{aligned} \tag{12.6}$$

$$\Rightarrow \mathrm{Re}\,h^E(\mathrm{Cl}(X)\sigma_1, \sigma_2) + \mathrm{Re}\,h^E(\sigma_1, \mathrm{Cl}(X)\sigma_2) = 0. \tag{12.7}$$

So substituting $-\mathfrak{i}\sigma_1$ for σ_1 we see that the $\mathrm{Cl}(X)$ are skew-symmetric.

$$h^E(\mathrm{Cl}(X)\sigma_1, \sigma_2) = -h^E(\sigma_1, \mathrm{Cl}(X)\sigma_2) \tag{12.8}$$

13. Lemma: Symmetry

D is symmetric in the L^2-scalar product on sections with compact support.

Proof With (e_i) being a local orthonormal frame of TM as in the definition of D and σ_1, σ_2 being smooth sections of E with compact support contained in the domain of (e_i), we calculate

$$\begin{aligned}\langle \mathrm{Cl}(e_i)\nabla^E_{e_i}\sigma_1, \sigma_2\rangle &\overset{(12.8)}{=} -\langle \nabla^E_{e_i}\sigma_1, \mathrm{Cl}(e_i)\sigma_2\rangle\\ &\overset{\nabla^E\text{ metric}}{=} \langle \sigma_1, \nabla^E_{e_i}\mathrm{Cl}(e_i)\sigma_2\rangle - e_i\langle\sigma_1, \mathrm{Cl}(e_i)\sigma_2\rangle\\ &\overset{(11.3)}{=} \langle\sigma_1, \mathrm{Cl}(e_i)\nabla^E_{e_i}\sigma_2\rangle + \langle\sigma_1, \mathrm{Cl}(\nabla_{e_i}e_i)\sigma_2\rangle - e_i\langle\sigma_1, \mathrm{Cl}(e_i)\sigma_2\rangle,\end{aligned} \tag{13.1}$$

so when integrating, we can apply the Leibniz rule for the divergence and the divergence theorem to get

$$\begin{aligned}\int_M e_i\langle\sigma_1, \mathrm{Cl}(e_i)\sigma_2\rangle\,\mathrm{vol}_g &= \int_M \mathrm{div}(\langle\sigma_1, \mathrm{Cl}(e_i)\sigma_2\rangle e_i) - \langle\sigma_1, \mathrm{Cl}(e_i)\sigma_2\rangle\,\mathrm{div}(e_i)\,\mathrm{vol}_g\\ &= -\int_M \langle\sigma_1, \mathrm{Cl}(e_i)\sigma_2\rangle\,\mathrm{div}(e_i)\,\mathrm{vol}_g\end{aligned} \tag{13.2}$$

$$\Rightarrow \int_M \langle D\sigma_1, \sigma_2\rangle \operatorname{vol}_g = \int_M \langle \sigma_1, D\sigma_2\rangle + \sum_{i=1}^m \langle \sigma_1, \mathrm{Cl}(\nabla_{e_i} e_i)\sigma_2 + \operatorname{div}(e_i)\,\mathrm{Cl}(e_i)\sigma_2\rangle \operatorname{vol}_g$$
$$= \int_M \langle \sigma_1, D\sigma_2 + \sum_{i=1}^m \mathrm{Cl}(\nabla_{e_i} e_i + \operatorname{div}(e_i) e_i)\sigma_2\rangle \operatorname{vol}_g. \tag{13.3}$$

Now the operator $\sum_{i=1}^m \mathrm{Cl}(\nabla_{e_i} e_i + \operatorname{div}(e_i)e_i)$ is tensorial. Fix a point and extent the orthonormal frame by parallel translation along geodesics. Then $\nabla_{e_i} e_i$, $\operatorname{div}(e_i)$ and so the whole term vanishes.

Compare also [Wol73, Proposition 4.3] or [LM89, Chapter II Proposition 5.3]. ◪

14. Theorem: Self-Adjointness

If (M, g) is a complete Riemannian Manifold, any Dirac operator associated to a Dirac bundle E over M is essentially self-adjoint.

Proof See [Wol73, Theorem 5.1] or [LM89, Chapter II Theorem 5.7]. ◪

15. Definition: "Restricted Dirac Bundle"

We define an adjusted Clifford multiplication on each $(E^t, h^t) := (E, h)|_{N^t}$ via

$$\mathrm{Cl}^t := \mathrm{Cl}(T) \circ \mathrm{Cl}|_{N^t} \tag{15.1}$$

and a Hermitian connection

$$\nabla^t_X := \nabla^E_X - \tfrac{1}{2}\,\mathrm{Cl}^t(W^t(X)), \quad X \in TN^t, \tag{15.2}$$

as in [BBC12, Equations (2.8) and (2.9)]. Here $W^t : TN^t \ni X \mapsto \nabla_X T \in TN^t$ is the *Weingarten operator* of N^t with respect to T.

16. Lemma:

Now E^t is a Dirac bundle over N^t, $\mathrm{Cl}(T)$ is ∇^t-parallel and

$$\nabla^t_X = \nabla^E_X - \tfrac{1}{2}\,\mathrm{Cl}(T)\,\mathrm{Cl}(\nabla_X T). \tag{16.1}$$

Write $\kappa^t := \operatorname{tr} W^t \in C^\infty(N^t, \mathbb{R})$ for (dim N times) the *mean curvature* of N^t in M_∞. Then the Dirac operator for this Dirac bundle, as in (11.4), takes the form

$$D^t\sigma = \sum_{i=2}^{\dim M} \mathrm{Cl}^t(e_i)(\nabla^t_{e_i}\sigma) = \sum_{i=2}^{\dim M} \mathrm{Cl}(T)\,\mathrm{Cl}(e_i)\nabla^E_{e_i}\sigma + \frac{\kappa^t}{2}\sigma, \tag{16.2}$$

if we complete T to a local orthonormal frame $(T, e_2, \ldots, e_{\dim N})$. Additionally we can write the Dirac operator on E over M_∞ as

$$D = \mathrm{Cl}(T)(\nabla^E_T - D^t + \frac{\kappa^t}{2}). \tag{16.3}$$

Proof Use (11.1) and (11.3) for parallelism of $\mathrm{Cl}(T)$. ☑

17. Remark: Sign

The above construction depends on the choice of T. Changing the sign of T will also change the sign of Cl^t, W, κ^t and D^t, while ∇^t and of course D remain invariant. Here we will always use $T = \operatorname{grad} f$ as in [BBC12, § 3.2]. This corresponds to "inward" as chosen in [BBC12, § 2.1] when considering N^t as boundary of $\{ p \in M \mid t \leq f(p) < \infty \}$, but is opposite to $\frac{\partial}{\partial u}$ in [APS74, § 2] because u is the boundary defining function x in our notation.

4. Vector Bundles and Differential Operators

Continuing to establish the situation of [BBC12], we turn to hermitian vector bundles and differential operators on them. Let E be any vector bundle over M_∞ with Hermitian metric h and metric connection ∇^E

In contrast to M_∞, E does not have a product structure over I. That is why we use parallel continuation to identify sections of E over the various N^t. In particular applying this, below defined, parallel continuation to Dirac bundles will give us a Dirac system just like in [BBC12, § 3.3].

At this point we need to choose some representative $N^{t_0} \cong N, t_0 \in I$ for the N^t as the source of the parallel transport. Then we separate the t-coordinate through an isomorphism $\mathrm{P}^E\colon C^\infty(I, \varGamma(N^{t_0}, E^{t_0})) \to \varGamma(M_\infty, E)$ with inverse $\P^E$, where $E^t := E|_{N^t}$.

18. Definition: "Parallel Continuation"

First of all, define P^E_t to be the *parallel continuation* from E^{t_0} to E^t along the geodesics in T-direction with inverse $\P^E_t\colon E^t \to E^{t_0}$. Secondly we can push forward the sections of E^{t_0} onto each of the E^t. This gives a map onto the T-parallel sections of E

$$\begin{aligned} &\mathrm{P}^E\colon \varGamma(N^{t_0}, E^{t_0}) \to \varGamma(M_\infty, E)\\ &\mathrm{P}^E\,\sigma\big|_{N^t} := \mathrm{P}^E_t \circ\sigma \qquad (18.1)\\ &\mathrm{P}^E\,\sigma\big|_{N^{t_0}} = \sigma \quad \wedge \quad \nabla^E_T\,\mathrm{P}^E\,\sigma = 0. \end{aligned}$$

Finally, just as in [BBC12, § 3.3], define P^E on $C^\infty(I, \varGamma(N^{t_0}, E^{t_0}))$ by

$$\mathrm{P}^E\,\sigma\big|_{N^t} = \mathrm{P}^E(\sigma(t)) \qquad (18.2)$$

and write $\P^E\colon \varGamma(M_\infty, E) \to C^\infty(I, \varGamma(N^{t_0}, E^{t_0}))$ for its inverse. P^E on $\varGamma(N^{t_0}, E^{t_0})$ can then be regarded as the special case where the element in $C^\infty(I, \varGamma(N^{t_0}, E^{t_0}))$ is constant.

Usually the bundle E needs not be stated explicitly, so we just write P and ¶.

19. Definition: "Vertical Differential Operators"

A differential operator $D \in \mathrm{Diff}(N, E)$ acting on a bundle E over a fibre bundle $(N, \phi, B; \mathcal{F})$ is called *vertical* if

$$\forall f \in C^\infty(B)\colon [D, \phi^* f] = 0. \qquad (19.1)$$

20. Lemma: First-Order Vertical Operators

A first-order differential operator D is vertical iff either of the following equivalent conditions are met:

1. $D \in \mathrm{span}_{\mathrm{End}(E}\left\{ \nabla^E_X \;\middle|\; X \in \varGamma(N, T^V N) \right\}$
2. The symbol $\hat{D}$ vanishes on $T^H N^*$.

Compare [Brü09, Equation (1.7)].

21. Lemma: Vertical Operators and Parallel Continuation

$D^t \in \mathrm{Diff}(N^t, E^t)$ is vertical iff $\P_t\, D^t\, \mathrm{P}_t \in \mathrm{Diff}(N^{t_0}, E^{t_0})$ is vertical.

Proof As $N^t = \{t\} \times N$ there is a canonical isomorphism $b^t = (t, \mathrm{id}_B)$ such that $(b^t)^*$ is an isomorphism between $C^\infty(B^t)$ and $C^\infty(B)$ which satisfies $(t, \phi^*(b^t)^* f) = (\phi^t)^* f$ for $f \in C^\infty(B^t)$. Furthermore P_t uses parallel transport along T-geodesics, which are of the form $t \mapsto (t, p), p \in N$, so multiplication by $(\phi^t)^* f$ on $\Gamma(N^{t_0}, E^{t_0})$ transforms as $\phi^*(b^t)^* f = \P_t \circ ((\phi^t)^* f) \circ \mathrm{P}_t$. From this we can conclude that

$$[\P_t\, D^t\, \mathrm{P}_t, \phi^*(b^t)^* f] = [\P_t\, D^t\, \mathrm{P}_t, \P_t (\phi^t)^* f\, \mathrm{P}_t] = \P_t [D^t, (\phi^t)^* f]\, \mathrm{P}_t \tag{21.1}$$

and the proof is complete, because $(b^t)^*$ is bijective. ☑

22. Definition: "L^2-spaces"

For (possibly non-continuous) sections σ_1 and σ_2 of E^t we have the usual L^2-scalar product

$$(\sigma_1, \sigma_2)_{L^2(E^t)} = \int_{N^t} h^t(\sigma_1, \sigma_2)\, \mathrm{vol}_{g^t}. \tag{22.1}$$

By using parallel transport, we can then define the t-scalar product for sections σ_1 and σ_2 of E^{t_0} as

$$(\sigma_1, \sigma_2)_t := (\mathrm{P}\,\sigma_1, \mathrm{P}\,\sigma_2)_{L^2(E^t)}. \tag{22.2}$$

Moreover we can define L^2-spaces for E over each fibre $\mathcal{F}_b := \phi^{-1}(b), b \in B$ as

$$(\sigma_1, \sigma_2)_{L^2(E|_{\mathcal{F}_b})} := \int_{\mathcal{F}_b} h^{t_0}(\sigma_1, \sigma_2)\, \mathrm{vol}_{\mathcal{F}_b}. \tag{22.3}$$

We will write $E_b := E|_{\mathcal{F}_b}$ for brevity.

23. Remark: t-Dependence in the Fibres

As calculated above, g^t does not depend on t for vertical tangent vectors. So neither does the corresponding volume form $\mathrm{vol}_{\mathcal{F}_b}$ and there is no need to define a t-scalar product on the fibres.

The scalar product over E^t can be calculated from the one on the fibres through

$$(\sigma_1, \sigma_2)_{L^2(E^t)} = \int_B (\sigma_1, \sigma_2)_{L^2(E_b)}\, \mathrm{vol}_{t^2 g_B}, \tag{23.1}$$

because by definition ϕ is a Riemannian submersion and so $\mathrm{vol}_{g^t} = \phi^*\, \mathrm{vol}_{t^2 g_B} \wedge \mathrm{vol}_{\mathcal{F}_b}$.

24. Remark: Convert Between t-Scalar Products

As P_t is an isometry between E^{t_0} and E^t, we have, with $F^t\colon (N^{t_0}, g^{t_0}) \ni (t_0, p) \mapsto (t, p) \in (N^t, g^t)$, that

$$\begin{aligned}(\sigma_1, \sigma_2)_t &= \int_{N^t} h^t(\mathrm{P}\,\sigma_1, \mathrm{P}\,\sigma_2)\,\mathrm{vol}_{g^t} \\ &= \int_N h^{t_0}(\sigma_1, \sigma_2)(\det \mathrm{d}F^t)\,\mathrm{vol}_{g^{t_0}} \\ &=: \int_N h^{t_0}(\sigma_1, \sigma_2) j(t)\,\mathrm{vol}_{g^{t_0}}.\end{aligned} \tag{24.1}$$

See [BBC12, Equation (3.38)].

For the metrics which we examine, $j(t)$ can be calculated explicitly. If we take a local frame $(\frac{t_0}{t}e_1, \ldots, \frac{t_0}{t}e_{\dim B}, e_{\dim B+1}, \ldots, e_{\dim N})$ as in Lemma 7, $\mathrm{d}F^t$ corresponds to multiplication by $\frac{t}{t_0}$ in the horizontal directions and identity in the vertical directions. This means that

$$j(t) = \det \mathrm{d}F^t = \left(\frac{t}{t_0}\right)^{\dim B} \tag{24.2}$$

and allows us to calculate the mean curvature

$$\kappa j = j' = \frac{1}{t}(\dim B)\left(\frac{t}{t_0}\right)^{\dim B} \tag{24.3}$$

$$\Rightarrow \kappa = (\dim B)\frac{1}{t}. \tag{24.4}$$

Thus we can confirm that κ does not depend on the choice of t_0, as expected.

25. Lemma: Transform Dirac Bundle into Dirac System

Let (E, h, Cl) be a Dirac bundle over M_∞ with Dirac operator D. Then $(\mathcal{H}, \mathcal{A}, \mathrm{Cl}\,T)$ is a Dirac system over I as defined in [BBC12, § 3.1] and [BBC08, § 2.1], where

$$\mathcal{H} = (H_t)_{t\in I} = (L^2(N^{t_0}, E^{t_0}), (\bullet, \bullet)_t)_t \tag{25.1}$$

$$\mathcal{A} = (A^t)_{t\in I} = (-\P\, D^t\, \mathrm{P})_t. \tag{25.2}$$

Proof This is exactly the situation of [BBC12, § 3.3]. ◪

26. Remark:

In the above setting, if X is a vector field on N^{t_0} then $\mathrm{P} \circ \mathrm{Cl}(X) = \mathrm{Cl}(\mathrm{P}^{TM}(X)) \circ \mathrm{P}$.

5. Spinors

In this part we define E as the spinor bundle $\mathcal{S}(M)$ over an even dimensional spin-manifold, show that it fulfils the necessary conditions to apply the methods from [BBC12] and prove an index formula like the one in [LMP06].

5.1. Geometry

Consider the *Clifford algebra* $\mathrm{Cl}_k, k \in \mathbb{N}$ over $\mathbb{R}^k$ with the standard metric. Denote the multiplication in Cl_k by $\cdot$ and write $\mathrm{Cl}_k = \mathrm{Cl}_k^+ \oplus \mathrm{Cl}_k^-$ for elements of even and odd degree. The inclusion $\mathbb{R}^k \subset \mathrm{Cl}_k$ will be kept implicit. For reference see [LM89] and [BGV92].

Let M be as before and of even dimension $m = 2n$ with a fixed spin structure ξ. The latter means that we have fixed an orientation on M, i.e. an SO_m-principal bundle $\mathrm{SO}(M) \subset \mathrm{O}(M)$ (where $\mathrm{O}(M)$ is defined by the metric), as well as a Spin_m-reduction $\xi\colon \mathrm{Spin}(M) \to \mathrm{SO}(M)$ with $\xi \circ \alpha = \mathrm{Ad}_m(\alpha) \circ \xi$ for $\alpha \in \mathrm{Spin}_m$, where $\mathrm{Ad}_m\colon \mathrm{Spin}_m \to \mathrm{SO}_m, \mathrm{Ad}_m(\alpha)(v) = \alpha \cdot v \cdot \alpha^{-1}$ is the adjoint representation. One consequence is that $TM \simeq \mathrm{Spin}(M) \times_{\mathrm{Ad}_m} \mathbb{R}^m$. (See [LM89, Chapter I § 2].)

27. Definition: "Complex Spinor Bundle"

Now take $E := \mathcal{S}(M)$ to be the *complex spinor bundle* associated to $\mathrm{Spin}(M)$ via the irreducible representation ρ of $\mathbb{C}\mathrm{l}_m := \mathrm{Cl}_m \otimes \mathbb{C}$ on $\mathcal{S}_m := \Lambda\mathbb{C}^n$, as defined in [BGV92, Proposition 3.19]. Secondly choose a Hermitian metric on $\mathcal{S}_m$ as in [BGV92, Proposition 3.26] and denote by h the induced metric on $\mathcal{S}(M)$. Finally pull up the Levi-Civita connection form on $\mathrm{SO}(M)$ to $\mathrm{Spin}(M)$ via ξ, to get an associated connection ∇^E on $\mathcal{S}(M)$.

Proof For $\sigma \in \mathcal{S}_m$ and $v \in \mathbb{R}^m, \|v\| = 1$ we have

$$v \cdot v = -1 \Rightarrow \|\rho(v)\sigma\|^2 = \langle \sigma, -\rho(v)\rho(v)\sigma \rangle = \|\sigma\|^2 \tag{27.1}$$

on $\mathcal{S}_m$. So as every element Spin_m can be expressed as a product of these v, Spin_m acts isometrically on $\mathcal{S}_m$ and h is well-defined by $h([(p,\sigma_1)],[(p,\sigma_2)]) := \langle \sigma_1, \sigma_2 \rangle_{\mathcal{S}_m}$ because

$$h([(p\cdot\alpha^{-1}, \rho(\alpha)(\sigma_1))],[(p\cdot\alpha^{-1}, \rho(\alpha)(\sigma_2))]) = \langle \rho(\alpha)(\sigma_1), \rho(\alpha)(\sigma_2) \rangle = \langle \sigma_1, \sigma_2 \rangle \tag{27.2}$$

for $p \in \mathrm{Spin}(M), \sigma_1, \sigma_2 \in \mathcal{S}_m$ and $\alpha \in \mathrm{Spin}_m$. ☑

28. Definition: "Clifford Multiplication"

Define the *Clifford multiplication* on $\mathcal{S}(M)$ through $\rho\colon \mathrm{Cl}_m \to \mathrm{End}(\mathcal{S}_m)$ on local frames, i.e.

$$\mathrm{Cl}([(p,v)])([(p,\sigma)]) := [(p, \rho(v)(\sigma))] \tag{28.1}$$

for $[(p,v)] \in TM$ and $[(p,\sigma)] \in \mathcal{S}(M)$ over the same base point in M and described by $p \in \mathrm{Spin}(M), v \in \mathbb{R}^m, \sigma \in \mathcal{S}_m$.

Proof For $\alpha \in \mathrm{Spin}_m$ we get

$$\begin{aligned}&\mathrm{Cl}([(p \cdot \alpha^{-1}, \mathrm{Ad}_m(\alpha)(v))])([(p \cdot \alpha^{-1}, \rho(\alpha)(\sigma))]) \\ &\quad = [(p \cdot \alpha^{-1}, \rho(\mathrm{Ad}_m(\alpha)(v))(\rho(\alpha)(\sigma)))] \\ &\quad = [(p \cdot \alpha^{-1}, \rho(\alpha)\rho(v)\rho(\alpha^{-1})\rho(\alpha)(\sigma))] = [(p, \rho(v)(\sigma))].\end{aligned} \tag{28.2}$$

So the notion is well-defined. ■

29. Lemma: Levi-Civita Connection

Take $\omega\colon \Omega^1(U, \mathfrak{so}_m)$ to be the local connection form of ∇ for the local orthonormal frame $\xi \circ p$, $p \in \Gamma(U, \mathrm{Spin}(M))$, $U \subset M$ (with $\mathfrak{so}_m$ being the Lie algebra of skew-symmetric $m \times m$-matrices). Then the Levi-Civita connection on $E = \mathcal{S}(M)$ in these frames is

$$\nabla^E = \mathrm{d} + \frac{1}{4} \sum_{j,k=1}^{m} \omega_{k,j} \rho(e_j \cdot e_k), \tag{29.1}$$

where e_i are the unit vectors in $\mathbb{R}^m$. Note that $\omega_{k,j}(e_i) = {}^{\phi}g(\nabla_{e_i} e_j, e_k) = \Gamma^k_{i,j}$.

Proof See [LM89, Chapter II Theorem 4.14] and [BGV92, Equation (3.13)]. ■

30. Lemma: Dirac Bundle

The complex spinor bundle $\mathcal{S}(M)$ is a Dirac bundle as in Definition 11.

Proof We saw above that ρ maps Spin_m into $\mathrm{O}(\mathcal{S}_m)$. So ∇^E is a metric connection. The same calculation used there gives, for arbitrary $v \in \mathbb{R}^m$, the local calculation

$$\begin{aligned}&|\mathrm{Cl}([(p, v)])([(p, \sigma)])|_E = |[(p, \rho(v)(\sigma))]|_E = |\rho(v)(\sigma)|_{\mathcal{S}_m} \\ &\quad = |v|_{\mathbb{R}^m} \cdot |\sigma|_{\mathcal{S}_m} = |[(p, v)]|_{TM} \cdot |[(p, \sigma)]|_E,\end{aligned} \tag{30.1}$$

with $p \in \mathrm{Spin}(M), \sigma \in \mathcal{S}_m$, because $\mathrm{Spin}(M)$ is a reduction of $\mathrm{SO}(M)$, which is associated to ${}^{\phi}g$. This shows (11.2). Analogue the Clifford relation $v \cdot v = -|v|^2$ immediately shows that (11.1) holds.

(11.3) is tensorial in Y, so it suffices to prove it for $Y = [(p, e_i)], i \in \{1, \ldots, m\}$. We know $\omega_{j,j} = 0$ and if $i \neq j \wedge i \neq k$ we have $e_i \cdot e_k \cdot e_j = e_k \cdot e_j \cdot e_i$ while $i = j \neq k \vee i = k \neq j$

implies $e_i \cdot e_k \cdot e_j = -e_k \cdot e_j \cdot e_i$, so we can calculate

$$\begin{aligned}
\nabla^E_X \operatorname{Cl}(Y)[(p,\sigma)] &= \nabla^E_X[(p,\rho(e_i)(\sigma))] = [(p, \tfrac{1}{4}\sum_{j,k=1}^m \omega_{j,k}(X)\rho(e_k \cdot e_j \cdot e_i)\sigma)] \\
&= [(p, \tfrac{1}{4}\sum_{j,k=1}^m \omega_{j,k}(X)\rho(e_i \cdot e_k \cdot e_j)\sigma)] + [(p, \tfrac{1}{2}\sum_{k=1}^m \omega_{i,k}(X)\rho(e_k \cdot e_i \cdot e_i)\sigma)] \\
&\quad + [(p, \tfrac{1}{2}\sum_{j=1}^m \omega_{j,i}(X)\rho(e_i \cdot e_j \cdot e_i)\sigma)] \\
&= \operatorname{Cl}(Y)\nabla^E_X[(p,\sigma)] + [(p, \sum_{j=1}^m \omega_{j,i}(X)\rho(e_j)\sigma)] \\
&= \operatorname{Cl}(Y)\nabla^E_X[(p,\sigma)] + \operatorname{Cl}(\nabla_X[(p,e_i)])[(p,\sigma)].
\end{aligned} \tag{30.2}$$

This shows (11.3) and completes the proof. ☑

31. Lemma: Graded Spinors

The spinor bundle is $\mathbb{Z}_2$-graded as $\mathcal{S}(M) = \mathcal{S}^+(M) \oplus \mathcal{S}^-(M)$ and the Clifford multiplication is odd in this grading. Thus the associated Dirac operator decomposes as

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} \tag{31.1}$$

where $D^- \colon \mathcal{S}^-(M) \to \mathcal{S}^+(M)$ is the adjoint of $D^+ \colon \mathcal{S}^+(M) \to \mathcal{S}^-(M)$.

Proof As shown in [BGV92, Proposition 3.19] the spinor module $\mathcal{S}_m$ is naturally graded as $\mathcal{S}_m^\pm := \Lambda^\pm \mathbb{C}^n$. Now because ρ is even on Spin_m it induces the stated grading on $\mathcal{S}(M)$.

The Clifford multiplication is also defined via ρ, but on elements of order 1, so they are odd. On the other hand we can see from Lemma 29 that the Levi-Civita connection is even and so the Dirac operator must be odd. Then we can simply define $D^+ := D|_{\mathcal{S}^+(M)}$ and the rest follows from the self-adjointness of D. ☑

32. Definition: "Embeddings"

The embedding $\mathbb{R}^{m-1} \ni v \mapsto (0,v) \in \mathbb{R}^m$ induces the homomorphisms

$$\iota_{\mathrm{SO}} \colon \mathrm{SO}_{m-1} \ni A \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & A \end{smallmatrix}\right) \in \mathrm{SO}_m \tag{32.1}$$

and

$$\iota_{\mathrm{Cl}} \colon \mathrm{Cl}_{m-1} \to \mathrm{Cl}_m, \tag{32.2}$$

which restricts to

$$\iota_{\mathrm{Spin}} \colon \mathrm{Spin}_{m-1} \to \mathrm{Spin}_m\,. \tag{32.3}$$

Then the normal unit vector field T gives us the SO_{m-1}-reduction

$$\tilde{T} \colon \mathrm{SO}(N) \ni (e_2,\dots,e_m) \mapsto (T, e_2, \dots, e_m) \in \mathrm{SO}(M). \tag{32.4}$$

33. Lemma: Spin Structure on N

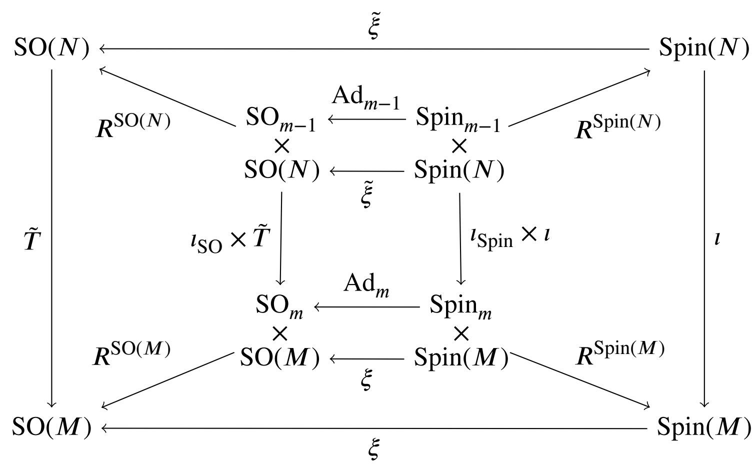
Spin(M) induces a spin structure $\tilde{\xi}\colon \mathrm{Spin}(N) \to \mathrm{SO}(N)$ on N.

Proof We define $\mathrm{Spin}(N) \coloneqq \xi^{-1}(\tilde{T}(\mathrm{SO}(N)))$, a subset of Spin($M$), denote the inclusion by $\iota\colon \mathrm{Spin}(N) \subset \mathrm{Spin}(M)$ and factorize ξ via $\tilde{\xi} \coloneqq \tilde{T}^{-1} \circ \xi$ on Spin(N). This is possible, because $\tilde{T}$ is injective. More specifically it is an injective embedding of smooth manifolds. As ξ is a smooth double cover, so is $\tilde{\xi}$ and Spin(N) is a submanifold of Spin(M). This also shows, that Spin(N) is a locally trivial fibre bundle over N, like SO(N).

Now we define the Spin_{m-1}-right action on Spin(N) as

$$R^{\mathrm{Spin}(N)} \coloneqq \iota^{-1} \circ R^{\mathrm{Spin}(M)} \circ (\iota_{\mathrm{Spin}} \times \iota). \tag{33.1}$$

In order to do this, let us look at the following diagram:



We already know that the left, inner and bottom subdiagrams commute, while the right and outer subdiagrams will commute by definition, but we first have to show that $R^{\mathrm{Spin}(N)}$ is well defined. For this purpose we start by applying the known commutativity properties to

$$\begin{aligned} &\xi \circ R^{\mathrm{Spin}(M)} \circ (\iota_{\mathrm{Spin}} \times \iota) \\ &\quad = R^{\mathrm{SO}(M)} \circ (\mathrm{Ad}_m \times \xi) \circ (\iota_{\mathrm{Spin}} \times \iota) \\ &\quad = R^{\mathrm{SO}(M)} \circ (\iota_{\mathrm{SO}} \times \tilde{T}) \circ (\mathrm{Ad}_{m-1} \times \tilde{\xi}) \\ &\quad = \tilde{T} \circ R^{\mathrm{SO}(N)} \circ (\mathrm{Ad}_{m-1} \times \tilde{\xi}). \end{aligned} \tag{33.2}$$

Then $R^{\mathrm{Spin}(N)}$ is well-defined, because

$$R^{\mathrm{Spin}(M)} \circ (\iota_{\mathrm{Spin}} \times \iota)(\mathrm{Spin}_{m-1} \times \mathrm{Spin}(N)) \subset \iota(\mathrm{Spin}(N)) \tag{33.3}$$

$$\Leftrightarrow \quad R^{\mathrm{Spin}(M)} \circ (\iota_{\mathrm{Spin}} \times \iota)(\mathrm{Spin}_{m-1} \times \xi^{-1}(\tilde{T}(\mathrm{SO}(N)))) \subset \xi^{-1}(\tilde{T}(\mathrm{SO}(N))) \tag{33.4}$$

$$\Leftrightarrow \quad \xi \circ R^{\mathrm{Spin}(M)} \circ (\iota_{\mathrm{Spin}} \times \iota)(\mathrm{Spin}_{m-1} \times \xi^{-1}(\tilde{T}(\mathrm{SO}(N)))) \subset \tilde{T}(\mathrm{SO}(N)) \tag{33.5}$$

$$\Leftrightarrow \quad \tilde{T} \circ R^{\mathrm{SO}(N)} \circ (\mathrm{Ad}_{m-1} \times \tilde{\xi})(\mathrm{Spin}_{m-1} \times \xi^{-1}(\tilde{T}(\mathrm{SO}(N)))) \subset \tilde{T}(\mathrm{SO}(N)) \tag{33.6}$$

$$\Leftrightarrow \quad R^{\mathrm{SO}(N)} \circ (\mathrm{Ad}_{m-1} \times \tilde{\xi})(\mathrm{Spin}_{m-1} \times \xi^{-1}(\tilde{T}(\mathrm{SO}(N)))) \subset \mathrm{SO}(N) \tag{33.7}$$

$$\Leftrightarrow \quad R^{\mathrm{SO}(N)} \circ (\mathrm{SO}_{m-1} \times \mathrm{SO}(N)) \subset \mathrm{SO}(N) \tag{33.8}$$

where we used for the last steps, that $\tilde{T}$ is injective as well as that ξ and Ad_{m-1} are surjective.

Having properly defined $R^{\mathrm{Spin}(N)}$, the commutativity of the right subdiagram follows from this definition and the commutativity of the top subdiagram is shown by taking the definition of $\tilde{\xi}$ and applying the commutativity of the right, bottom, inner and left subdiagrams in order (which comprises the steps in (33.2)).

Together with the smoothness of $R^{\mathrm{Spin}(N)}$, which is clear from its definition, this shows, that $\mathrm{Spin}(N)$ is a Spin_{m-1}-principle bundle over N and an Ad_{m-1}-reduction of $\mathrm{SO}(N)$. Moreover $\mathrm{Spin}(N)$ is a Spin_{m-1}-reduction of $\mathrm{Spin}(M)|_N$. ☑

34. Definition: "Adjusted Representation"

Define the algebra homomorphism

$$\tilde{\iota}_{\mathrm{Cl}} \colon \mathrm{Cl}^+_{m-1} \oplus \mathrm{Cl}^-_{m-1} \ni \alpha^+ + \alpha^- \mapsto \alpha^+ + e_1 \cdot \alpha^- \in \mathrm{Cl}^+_m \subset \mathrm{Cl}_m \tag{34.1}$$

with e_1 being the first unit vector in $\mathbb{R}^m$. It restricts to $\iota_{\mathrm{Spin}} \colon \mathrm{Spin}_{m-1} \to \mathrm{Spin}_m$ as defined before. Then we can define the representation

$$\tilde{\rho} := \rho \circ \tilde{\iota}_{\mathrm{Cl}} \colon \mathrm{Cl}_{m-1} \to \mathrm{End}(\mathcal{S}^+_m) \oplus \mathrm{End}(\mathcal{S}^-_m) \subseteq \mathrm{End}(\mathcal{S}_m). \tag{34.2}$$

35. Lemma:

The spinor bundle $\mathcal{S}(N) := \mathrm{Spin}(N) \times_{\tilde{\rho}} \mathcal{S}_m$ is isomorphic to $\mathcal{S}(M)|_N = E^{t_0}$ and its natural Clifford multiplication and Levi-Civita connection are those of Definition 15.

Proof $\mathcal{S}(N) \simeq E^{t_0}$ follows from the facts that $\mathrm{Spin}(N)$ is a reduction of $\mathrm{Spin}(M)|_N$ and that $\tilde{\rho} = \rho \circ \iota_{\mathrm{Spin}}$ over Spin_{m-1}.

By *natural Clifford multiplication* we mean the definition analogous to Definition 28 using $\tilde{\rho}$. Thus the definition of $\tilde{\iota}_{\mathrm{Cl}}$ for elements of degree one shows that this Clifford multiplication is exactly $\mathrm{Cl}(T) \circ \mathrm{Cl}|_N = \mathrm{Cl}^{t_0}$.

The Levi-Civita connection of $\mathcal{S}(N)$ in local coordinates is according to Lemma 29

$$\begin{aligned}\nabla^{\mathcal{S}(N)} &= \mathrm{d} + \tfrac{1}{4}\sum_{j,k=2}^{m} \omega_{k,j}\tilde{\rho}(e_j \cdot e_k) = \nabla^{\mathcal{S}(M)} - 2 \cdot \tfrac{1}{4}\sum_{k=2}^{m} \omega_{k,1}\rho(e_1 \cdot e_k) \\ &= \nabla^{\mathcal{S}(M)} - \tfrac{1}{2}\,\mathrm{Cl}(T)\,\mathrm{Cl}(\sum_{k=2}^{m} \omega_{k,1} e_k) = \nabla^{\mathcal{S}(M)} - \tfrac{1}{2}\,\mathrm{Cl}(T)\,\mathrm{Cl}(\nabla T) = \nabla^{t_0}.\end{aligned} \tag{35.1}$$

So ∇^{t_0} is the Levi-Civita connection on $\mathcal{S}(N)$. ☑

36. Remark:

$\rho \circ \iota_{\mathrm{Cl}}$ and $\tilde{\rho}$ coincide on Spin_{m-1}, thus yield the same spinor bundle and Levi-Civita connection, but the associated Clifford multiplications are different and for the former it does not fulfil the precondition for a Dirac bundle in (11.3).

5.2. Dirac Operator

Let $\mathcal{D}$ be the Dirac system as defined in Lemma 25 for the Dirac bundle $E = \mathcal{S}(M_\infty)$. In order to show that the Dirac operator D of $\mathcal{D}$ is non-parabolic, we will explicitly calculate the t-dependence in the family of self-adjoint operators $-A^t = \text{ꟼ}_t\, D^t\, \mathrm{P}_t$. For that purpose we decompose A^t into its vertical and horizontal parts as was also done in [Brü09, Section 1].

37. Definition: "Parallel Spinor Frames"

Start by taking the local orthonormal frame $(T, \bar{e}_2, \ldots, \bar{e}_m)$ of $TM_\infty|_{I\times U}$ as in Lemma 7. Then lift it up to a local section p of the principle fibre bundle $\mathrm{Spin}(M_\infty)$. This yields an associated local spinor frame $s\colon E|_{I\times U} \to U \times \mathbb{C}^{2^n}$.

As ∇^E is defined by pulling up ω from $\mathrm{SO}(M_\infty)$ to $\mathrm{Spin}(M_\infty)$ and taking the associated connection on E, we know that p and s are still parallel in T direction. In particular $s = s^{t_0}\,\text{ꟼ}$ for $s^{t_0} := s|_{\text{ẋè}U}$.

38. Theorem: Decomposition of the Dirac Operator

There is a vertical differential operator of first order A^∞_V and a first order differential operator A^∞_H as well as a symmetric endomorphism $\beta_2^{t_0}$ such that the t-dependence of the Dirac operator can be written as follows

$$-A^t = A^\infty_V + \tfrac{t_0}{t} A^\infty_H - \tfrac{t_0{}^2}{t^2}\beta_2^{t_0}. \tag{38.1}$$

Proof For $\sigma \in \Gamma(N, E^{t_0})$ and $e_i \in TN$

$$\text{ꟼ}\, s^{-1}\mathrm{d}(s\,\mathrm{P}\,\sigma)(\bar{e}_i) = (s^{t_0})^{-1}\mathrm{d}(s^{t_0}\sigma)(\bar{e}_i) = \begin{cases} (s^{t_0})^{-1}\mathrm{d}(s^{t_0}\sigma)(e_i) & e_i \in T^V N \\ \tfrac{t_0}{t}(s^{t_0})^{-1}\mathrm{d}(s^{t_0}\sigma)(e_i) & e_i \in T^H N. \end{cases} \tag{38.2}$$

So we will do our further calculation in these frames s^{t_0} and write d instead of $(s^{t_0})^{-1}\mathrm{d}(s^{t_0}\,\bullet)$.

Continuing, according to (11.3),

$$\text{ꟼ}\,\mathrm{Cl}^t(\bar{e}_i)\,\mathrm{P} = \text{ꟼ}\,\mathrm{Cl}(T)\,\mathrm{Cl}(\bar{e}_i)\,\mathrm{P} = \mathrm{Cl}(T)\,\mathrm{Cl}(e_i) = \mathrm{Cl}^{t_0}(e_i). \tag{38.3}$$

To ease the notation we will write $c_i^\bullet := \tilde{\rho}(e_i^\bullet)$. We use Lemma 7 to calculate in local frames

$$\begin{aligned} \text{ꟼ}\,\nabla^t_{\bar{e}_i^H}\,\mathrm{P} &= \tfrac{t_0}{t}\mathrm{d}\bullet(e_i^H) + \tfrac{1}{4}\sum_{j,k=2}^{m} \Gamma^k_{i,j}(t) c_j c_k \\ &= \tfrac{t_0}{t}\mathrm{d}\bullet(e_i^H) + \tfrac{1}{4}\cdot\tfrac{t_0}{t}\sum_{j,k \text{ hor.}} \Gamma^k_{i,j}(t_0) c_j^H c_k^H + \tfrac{1}{4}\cdot\tfrac{t_0}{t}\sum_{j,k \text{ vert.}} \Gamma^k_{i,j}(t_0) c_j^V c_k^V \\ &\quad + \tfrac{1}{4}\cdot\tfrac{t_0{}^2}{t^2}\sum_{\substack{j \text{ hor.} \\ k \text{ vert.}}} g^{t_0}(e_k^V, [e_i^H, e_j^H]) c_j^H c_k^V \end{aligned} \tag{38.4}$$

$$\text{ꟼ}\,\nabla^t_{\bar{e}_i^V}\,\mathrm{P} = \mathrm{d}\bullet(e_i^V) + \frac{1}{4}\sum_{j,k \text{ vert.}} \Gamma^k_{i,j}(t_0)c_j^V c_k^V + \frac{1}{2}\cdot\frac{t_0}{t}\sum_{\substack{j \text{ hor.}\\ k \text{ vert.}}} \Gamma^k_{i,j}(t_0)c_j^H c_k^V$$
$$- \frac{1}{8}\cdot\frac{{t_0}^2}{t^2}\sum_{j,k \text{ hor.}} g^{t_0}(e_i^V,[e_j^H,e_k^H])c_j^H c_k^H \tag{38.5}$$

In [Brü09, Section 1] Brüning symmetrizes the vertical and horizontal parts of the naive decomposition of the Dirac operator. When applying this method to the Dirac bundles $E^t \to N^t$ with Dirac operator D^t we get the following formulas.

$$\begin{aligned}
\tilde{A}^t_H &:= \sum_{i \text{ hor.}} c_i^H\,\text{ꟼ}\,\nabla^t_{\bar{e}_i^H}\,\mathrm{P}\\
&= \frac{t_0}{t}\sum_{i \text{ hor.}} c_i^H \mathrm{d}\bullet(e_i^H) + \frac{1}{4}\cdot\frac{t_0}{t}\sum_{i,j,k \text{ hor.}} \Gamma^k_{i,j}(t_0)c_i^H c_j^H c_k^H + \frac{1}{4}\cdot\frac{t_0}{t}\sum_{\substack{i \text{ hor.}\\ j,k \text{ vert.}}} \Gamma^k_{i,j}(t_0)c_i^H c_j^V c_k^V\\
&\quad + \frac{1}{4}\cdot\frac{{t_0}^2}{t^2}\sum_{\substack{i,j \text{ hor.}\\ k \text{ vert.}}} g^{t_0}(e_k^V,[e_i^H,e_j^H])c_i^H c_j^H c_k^V
\end{aligned} \tag{38.6}$$

$$\begin{aligned}
\tilde{A}^t_V &:= \sum_{i \text{ vert.}} c_i^V\,\text{ꟼ}\,\nabla^t_{\bar{e}_i^V}\,\mathrm{P}\\
&= \sum_{i \text{ vert.}} c_i^V \mathrm{d}\bullet(e_i^V) + \frac{1}{4}\sum_{i,j,k \text{ vert.}} \Gamma^k_{i,j}(t_0)c_i^V c_j^V c_k^V + \frac{1}{2}\cdot\frac{t_0}{t}\sum_{\substack{j \text{ hor.}\\ i,k \text{ vert.}}} \Gamma^k_{i,j}(t_0)c_i^V c_j^H c_k^V\\
&\quad - \frac{1}{8}\cdot\frac{{t_0}^2}{t^2}\sum_{\substack{j,k \text{ hor.}\\ i \text{ vert.}}} g^{t_0}(e_i^V,[e_j^H,e_k^H])c_i^V c_j^H c_k^H
\end{aligned} \tag{38.7}$$

$$\beta_1^t := \sum_{\substack{j \text{ hor.}\\ i \text{ vert.}}} \Gamma^j_{i,i}(t)c_j^H = \frac{t_0}{t}\sum_{\substack{j \text{ hor.}\\ i \text{ vert.}}} \Gamma^j_{i,i}(t_0)c_j^H \overset{(8.2)}{=} \frac{t_0}{t}\sum_{\substack{j \text{ hor.}\\ i,k \text{ vert.}}} \Gamma^j_{i,k}(t_0)c_i^V c_j^H c_k^V \tag{38.8}$$

$$\begin{aligned}
\beta_2^t &:= \frac{1}{4}\sum_{\substack{i,k \text{ hor.}\\ j \text{ vert.}}} \Gamma^j_{i,k}(t)c_j^V c_k^H c_i^H = \frac{1}{8}\sum_{\substack{i,k \text{ hor.}\\ j \text{ vert.}}} \frac{{t_0}^2}{t^2} g^{t_0}(e_j^V,[e_i^H,e_k^H])c_j^V c_k^H c_i^H\\
&= \frac{1}{8}\cdot\frac{{t_0}^2}{t^2}\sum_{\substack{i,j \text{ hor.}\\ k \text{ vert.}}} g^{t_0}(e_k^V,[e_i^H,e_j^H])c_k^V c_j^H c_i^H = -\frac{1}{8}\cdot\frac{{t_0}^2}{t^2}\sum_{\substack{i,j \text{ hor.}\\ k \text{ vert.}}} g^{t_0}(e_k^V,[e_i^H,e_j^H])c_i^H c_j^H c_k^V\\
&= \frac{1}{8}\cdot\frac{{t_0}^2}{t^2}\sum_{\substack{j,k \text{ hor.}\\ i \text{ vert.}}} g^{t_0}(e_i^V,[e_j^H,e_k^H])c_i^V c_k^H c_j^H = -\frac{1}{8}\cdot\frac{{t_0}^2}{t^2}\sum_{\substack{j,k \text{ hor.}\\ i \text{ vert.}}} g^{t_0}(e_i^V,[e_j^H,e_k^H])c_i^V c_j^H c_k^H\\
&= \frac{{t_0}^2}{t^2}\beta_2^{t_0}
\end{aligned} \tag{38.9}$$

This gives the decomposition $-A^t = A^t_H + A^t_V$ with

$$\begin{aligned}
A^t_H &:= \tilde{A}^t_H - \tfrac{1}{2}\beta^t_1 \\
&= \tfrac{t_0}{t}\sum_{i \text{ hor.}} c^H_i \mathrm{d}\bullet(e^H_i) + \tfrac{1}{4}\cdot\tfrac{t_0}{t}\sum_{i,j,k \text{ hor.}} \Gamma^k_{i,j}(t_0) c^H_i c^H_j c^H_k + \tfrac{1}{4}\cdot\tfrac{t_0}{t}\sum_{\substack{i \text{ hor.}\\ j,k \text{ vert.}}} \Gamma^k_{i,j}(t_0) c^H_i c^V_j c^V_k \\
&\quad - \tfrac{1}{2}\cdot\tfrac{t_0}{t}\sum_{\substack{j \text{ hor.}\\ i,k \text{ vert.}}} \Gamma^j_{i,k}(t_0) c^V_i c^H_j c^V_k + \tfrac{1}{4}\cdot\tfrac{t_0^2}{t^2}\sum_{\substack{i,j \text{ hor.}\\ k \text{ vert.}}} g^{t_0}(e^V_k,[e^H_i,e^H_j]) c^H_i c^H_j c^V_k \\
&= \tfrac{t_0}{t}(A^{t_0}_H + 2\beta^{t_0}_2) - 2\cdot\tfrac{t_0^2}{t^2}\beta^{t_0}_2 =: \tfrac{t_0}{t}A^\infty_H - 2\cdot\tfrac{t_0^2}{t^2}\beta^{t_0}_2
\end{aligned} \tag{38.10}$$

$$\begin{aligned}
A^t_V &:= \tilde{A}^t_V + \tfrac{1}{2}\beta^t_1 \\
&= \sum_{i \text{ vert.}} c^V_i \mathrm{d}\bullet(e^V_i) + \tfrac{1}{4}\sum_{i,j,k \text{ vert.}} \Gamma^k_{i,j}(t_0) c^V_i c^V_j c^V_k - \tfrac{1}{8}\cdot\tfrac{t_0^2}{t^2}\sum_{\substack{j,k \text{ hor.}\\ i \text{ vert.}}} g^{t_0}(e^V_i,[e^H_j,e^H_k]) c^V_i c^H_j c^H_k \\
&= (A^{t_0}_V - \beta^{t_0}_2) + \tfrac{t_0^2}{t^2}\beta^{t_0}_2 =: A^\infty_V + \tfrac{t_0^2}{t^2}\beta^{t_0}_2
\end{aligned} \tag{38.11}$$

and so we get the result

$$\text{P}\, D^t\, \text{P} = -A^t = A^t_V + A^t_H = A^\infty_V + \tfrac{t_0}{t}A^\infty_H - \tfrac{t_0^2}{t^2}\beta^{t_0}_2 \tag{38.12}$$

as asserted. ◪

39. Corollary:

We can immediately calculate, because $\frac{d}{dt}$ commutes with A^∞_H and A^∞_V, that

$$A' := [\frac{d}{dt}, A^t] = \tfrac{t_0}{t^2}A^\infty_H - 2\cdot\tfrac{t_0^2}{t^3}\beta^{t_0}_2 = \tfrac{1}{t}A^t_H. \tag{39.1}$$

40. Remark: Dependence on the Interval

From the definitions in [BBC12, § 3.2 and § 3.3] it is clear, that the theory of distance functions and Dirac systems is invariant under shifts of the interval. So, in particular, if $\mathcal{D}$ is a Dirac system over $[t_0, \infty)$, by setting $\tilde{t} := t - t_0$ we get a Dirac system $\tilde{\mathcal{D}}$ over $[0, \infty)$. The results of [BBC12, § 4] for $\tilde{\mathcal{D}}$ will then immediately carry over to $\mathcal{D}$.

Furthermore when defining the Dirac system starting from a manifold with straight ends, we can always restrict the distance function to a smaller interval $[t_1, \infty) \subset [t_0, \infty)$.

41. Lemma: Dirac Operator Estimate

For any closable, first order, vertical differential operator $\tilde{A}_{HV}$ on E^{t_0} there is a constant $c > 0$ such that

$$|\tilde{A}_{HV}| \le c(1 + |A^\infty_V|) \tag{41.1}$$

with respect to the t-scalar product for any $t \in I$.

Compare also [Brü09, Theorem 2.5].

Proof We know from [Brü09, Theorem 1.3] that $A_V^{t_0}$ is the Dirac operator of the Dirac bundle E_b with the Hermitian metric restricted from E and an adjusted connection $\nabla^{E,b}$. Write $\nabla^b := \nabla^{E,b}$ and denote by $\nabla^{b,*}$ its formal adjoint. In this proof we follow the argument used in [Fri00, Section 4.2 pp. 100-102]. It is sufficient to prove the estimate for smooth sections as $\Gamma(E)$ is dense in $L^2(E)$.

First of all for $b \in B$ and $\sigma \in \Gamma(E_b)$ the vertical Sobolev norm is defined as

$$\|\sigma\|^2_{H_1(E_b)} := \|\sigma\|^2_{L^2(E_b)} + \|\nabla^b\sigma\|^2_{L^2(T^*\mathcal{F}_b\otimes E_b)}. \tag{41.2}$$

Estimate of the Vertical Sobolev Norm The Lichnerowicz formula states

$$(A_V^{t_0})^2 = \nabla^{b,*}\nabla^b + R \tag{41.3}$$

with curvature terms $R \in \mathrm{End}(E_b)$ (compare [LM89, Section II Theorem 8.8] and [BGV92, Theorem 3.52]). Then using the symmetry of $A_V^{t_0}$, we get

$$\|A_V^{t_0}\sigma\|^2_{L^2(E_b)} = \|\nabla^b\sigma\|^2_{L^2(T^*M\otimes E_b)} + (R\sigma,\sigma)_{L^2(E_b)} \tag{41.4}$$

$$\Rightarrow \|\nabla^b\sigma\|^2_{L^2(T^*M\otimes E_b)} \le \|A_V^{t_0}\sigma\|^2_{L^2(E_b)} + \sup_{x\in\mathcal{F}_b}\|R\|_{\mathrm{End}(E|_x)}\|\sigma\|^2_{L^2(E_b)} \tag{41.5}$$

$$\Rightarrow \|\sigma\|^2_{H^1(E_b)} \le c_1(\|\sigma\|^2_{L^2(E_b)} + \|A_V^{t_0}\sigma\|^2_{L^2(E_b)}) \tag{41.6}$$

for some $c_1 > 0$, because $\mathcal{F}_b$ is compact. Moreover B is also compact and so c_1 can be chosen independently of b.

t-Norm Estimate Because $\tilde{A}_{HV}$ is a first order, vertical differential operator we have $c_2 > 0$ such that

$$\begin{aligned}\|\tilde{A}_{HV}\sigma\|^2_{L^2(E_b)} \le c_2\|\sigma\|^2_{H^1(E_b)} &\le c_1c_2(\|\sigma\|^2_{L^2(E_b)} + \|A_V^{t_0}\sigma\|^2_{L^2(E_b)})\\ &= c_1c_2(\|\sigma\|^2_{L^2(E_b)} + \|(A_V^\infty + \beta_2^{t_0})\sigma\|^2_{L^2(E_b)})\\ &\le c_1c_2(\|\sigma\|^2_{L^2(E_b)} + \|A_V^\infty\sigma\|^2_{L^2(E_b)} + \|\beta_2^{t_0}\sigma\|^2_{L^2(E_b)})\\ &\le c_3(\|\sigma\|^2_{L^2(E_b)} + \|A_V^\infty\sigma\|^2_{L^2(E_b)}).\end{aligned} \tag{41.7}$$

c_2 can also be chosen independently of b, because $\tilde{A}_{HV}$ is a differential operator with continuous coefficients and B is compact. Now we can estimate the t-norm

$$\begin{aligned}\|\tilde{A}_{HV}\sigma\|^2_t &= \int_B \|\tilde{A}_{HV}\sigma\|^2_{L^2(E_b)}\,\mathrm{vol}_{t^2g_B}\\ &\le \int_B c_3(\|\sigma\|^2_{L^2(E_b)} + \|A_V^\infty\sigma\|_{L^2(E_b)})\,\mathrm{vol}_{t^2g_B} = c_3(\|\sigma\|^2_t + \|A_V^\infty\sigma\|^2_t).\end{aligned} \tag{41.8}$$

Operator Inequality The norm estimate gives us

$$(|\tilde{A}_{HV}|\sigma,\sigma)_t \le c(\sqrt{1+(A_V^\infty)^* A_V^\infty}\sigma,\sigma)_t \le c((1+|A_V^\infty|)\sigma,\sigma)_t \tag{41.9}$$
$$\Rightarrow |\tilde{A}_{HV}| \le c(1+|A_V^\infty|) \tag{41.10}$$

with respect to the t-scalar product, as the square root is a complete Bernstein function. ◪

42. Theorem: Fredholm type

If the boundary Dirac operator A_V^∞ is invertible, $\mathcal{D}$, restricted to $[\tau(t_0),\infty)$ for $\tau(t_0)\ge t_0$ sufficiently big, is non-parabolic and it is even of Fredholm type as defined in [BBC12, § 4.3].

Proof We begin by calculating

$$\begin{aligned}\tilde{A}^t_{HV} &:= \left\{A_H^t+\beta_2^t, A_V^t-\beta_2^t\right\} = \left\{\tfrac{t_0}{t}A_H^\infty - \tfrac{t_0^2}{t^2}\beta_2^{t_0}, A_V^\infty\right\}\\ &= \tfrac{t_0}{t}\left\{A_H^\infty, A_V^\infty\right\} - \tfrac{t_0^2}{t^2}\left\{\beta_2^{t_0}, A_V^\infty\right\},\end{aligned} \tag{42.1}$$

where both summands are symmetric. According to [Brü09, Theorem 1.2] we know that $\tilde{A}^t_{HV}$ is first-order vertical (compare the calculation in Appendix A). As this holds for all $t\in I$, the operators $\left\{A_H^\infty, A_V^\infty\right\}$ and $\left\{\beta_2^{t_0}, A_V^\infty\right\}$ are first-order vertical as well.

Thus we can apply the preceding lemma to the operators $\left\{A_H^\infty, A_V^\infty\right\}$ and $\left\{\beta_2^{t_0}, A_V^\infty\right\}$ which yields

$$|\left\{A_H^\infty, A_V^\infty\right\}| \le c_1(1+|A_V^\infty|) \text{ and} \tag{42.2}$$
$$|\left\{\beta_2^{t_0}, A_V^\infty\right\}| \le c_2(1+|A_V^\infty|) \tag{42.3}$$

for fixed $c_1, c_2 > 0$. Then we can apply these inequalities to get the following estimate

$$\begin{aligned}(A^t)^2 - A' &= ((A_H^t+\beta_2^t)+(A_V^t-\beta_2^t))^2 - \tfrac{1}{t}A_H^t\\ &= (A_V^\infty)^2 + \tilde{A}^t_{HV} + (A_H^t+\beta_2^t)^2 - \tfrac{1}{t}A_H^t\\ &= (A_V^\infty)^2 + \tfrac{t_0}{t}\left\{A_H^\infty, A_V^\infty\right\} - \tfrac{t_0^2}{t^2}\left\{\beta_2^{t_0}, A_V^\infty\right\}\\ &\quad + (A_H^t+\beta_2^t - \tfrac{1}{2}\cdot\tfrac{1}{t})^2 + \tfrac{1}{t}\beta_2^t - \tfrac{1}{4}\cdot\tfrac{1}{t^2}\\ &\ge (A_V^\infty)^2 - c_1\tfrac{t_0}{t}(1+|A_V^\infty|) - c_2\tfrac{t_0^2}{t^2}(1+|A_V^\infty|) - \tfrac{t_0^2}{t^3}\|\beta_2^{t_0}\| - \tfrac{1}{4}\cdot\tfrac{1}{t^2}\\ &= (|A_V^\infty| - \tfrac{1}{2}(c_1\tfrac{t_0}{t}+c_2\tfrac{t_0^2}{t^2}))^2 - \tfrac{1}{4}(c_1\tfrac{t_0}{t}+c_2\tfrac{t_0^2}{t^2})^2\\ &\quad - c_1\tfrac{t_0}{t} - c_2\tfrac{t_0^2}{t^2} - \tfrac{1}{4}\cdot\tfrac{1}{t^2} - \tfrac{t_0^2}{t^3}\|\beta_2^{t_0}\|.\end{aligned} \tag{42.4}$$

Now because A_V^∞ is invertible $|A_V^\infty|$ is bounded below by its smallest positive eigenvalue $\lambda_{\min}$ and for any $\varepsilon, 0 < \varepsilon < \lambda_{\min}$ and t bigger than some sufficiently large t_ε we have

$$(A^t)^2 - A' \geq (\lambda_{\min} - \varepsilon)^2 - \varepsilon =: a > 0. \tag{42.5}$$

This is exactly the requirement of [BBC12, Proposition 4.40] and as our $a > 0$ we even have that D is of Fredholm type. ☑

43. Corollary: More Useful Estimates

Using the same technique as in the previous proof, we can show that for sufficiently large $\tau(t_0)$ and $t \geq \tau(t_0)$

- A^t is invertible and
- the preconditions for [BBC12, Proposition 4.46] are satisfied

if A_V^∞ is invertible.

Proof The first statement is obvious because instead of $(A^t)^2 - A'$ be can quite easily estimate $(A^t)^2$.

For the second statement we need to show that

$$c_0 := \sup_{\substack{t \geq \tau(t_0) \\ \sigma \in H_A \setminus \{0\}}} \frac{\|(A')^t \sigma\|_t}{\|\sigma\|_t + \|A^t \sigma\|_t} \tag{43.1}$$

can be made sufficiently small, so that there is a Λ inside the spectral gap of all A^t for which

$$\begin{aligned} & \Lambda > 2c_0 + \sqrt{8c_0(c_0 + 1)} \\ \Rightarrow \quad & \Lambda^2 > 4c_0(c_0 + 2 + \Lambda) \end{aligned} \tag{43.2}$$

holds, while for λ we can conveniently choose 0.

We write c_ε for this sufficiently small c_0. Then we need to show that for t sufficiently big and $\sigma \in H_A \setminus \{0\}$

$$\begin{aligned} & ((A')^t, (A')^t)_t \leq c_\varepsilon((\sigma, \sigma)_t + (A_t\sigma, A_t\sigma)_t) \\ \Leftrightarrow \quad & ((\tfrac{1}{t} A_H^t)^2 \sigma, \sigma)_t \leq c_\varepsilon((\sigma, \sigma)_t + ((A^t)^2\sigma, \sigma)_t) \\ \Leftrightarrow \quad & \frac{1}{c_\varepsilon t^2}(A_H^t)^2 \leq 1 + (A^t)^2 \end{aligned} \tag{43.3}$$

Now this is again quite similar to the previous estimate of $(A^t)^2 - A'$.

$$
\begin{aligned}
&(A^t)^2 - \frac{1}{c_\varepsilon t^2}(A^t_H)^2 + 1 \\
&\geq (A^\infty_V)^2 - c_1 \tfrac{t_0}{t}(1 + |A^\infty_V|) - c_2 \tfrac{t_0{}^2}{t^2}(1 + |A^\infty_V|) + (A^t_H + \beta^t_2)^2 - \frac{1}{c_\varepsilon t^2}(A^t_H)^2 + 1 \\
&= (|A^\infty_V| - \tfrac{1}{2}(c_1 \tfrac{t_0}{t} + c_2 \tfrac{t_0{}^2}{t^2}))^2 - \tfrac{1}{4}(c_1 \tfrac{t_0}{t} + c_2 \tfrac{t_0{}^2}{t^2})^2 - c_1 \tfrac{t_0}{t} - c_2 \tfrac{t_0{}^2}{t^2} \\
&\quad + (\sqrt{1 - \frac{1}{c_\varepsilon t^2}} A^t_H + \frac{1}{\sqrt{1 - \frac{1}{c_\varepsilon t^2}}} \beta^t_2)^2 - (\frac{1}{1 - c_\varepsilon t^2} - 1)(\beta^t_2)^2 + 1 \\
&\geq (\lambda_{\min} - \varepsilon)^2 - \varepsilon + 1
\end{aligned}
\tag{43.4}
$$

☑

5.3. Index Formula

We now consider the decomposition introduced in [BBC12, § 5]. There M is decomposed by some level surface N into a compact part M_0 and the ends U_0 such that $M = M_0 \cup U_0$ and $N = M_0 \cap U_0$. Using the previously established notation, we choose $N = N^\tau$ as a level surface of the distance function f defined in (3.1). As we will consider different values of τ and in particular have τ approach $+\infty$, we denote the decomposition by

$$
M = M^\tau \cup U^\tau, \qquad N^\tau = M^\tau \cap U^\tau \tag{5-1}
$$

It is important that τ is sufficiently large, i.e. $\tau \geq \tau(t_0)$ for some $t_0 \in I$ examined in the previous section.

Now [BBC12, Theorem 5.13 and Equation (5.21)] form the basis for our index formula. In order to use [BBC12, Equation (5.21)] we need to transform our metric ${}^\phi g$ near $N \subset M^\tau$ into a cylindrical metric.

44. Definition: "APS-Connection"

Using the diffeomorphism F from Definition 4 between M_∞ and $U \supset N^\tau$ we can extend ${}^\phi g|_{N^\tau}$ constantly inwards on $M^\tau \cap U$ as ${}^\phi g^\tau$. Next we take a smooth cut-off function $\varphi \colon M^\tau \to \mathbb{R}_+$ with $\varphi \equiv 1$ near N^τ and $\varphi \equiv 0$ outside some larger open neighbourhood. Now define the metric ${}^\tau\tilde{g}$ on the whole of M^τ through ${}^\tau\tilde{g} := \varphi \cdot {}^\phi g^\tau + (1 - \varphi) \cdot {}^\phi g$.

Then $\mathcal{S}(M)|_{M^\tau}$ over $(M^\tau, {}^\tau\tilde{g})$ is a spinor bundle to which the APS-Theorem applies. By ${}^\tau\tilde{\nabla}$ we denote the corresponding Levi-Civita connection and by ${}^\tau\tilde{\omega}$ its connection form.

45. Definition: "Transgression"

Consider the linear interpolation from ∇ to ${}^\tau\tilde{\nabla}$ denoted by ${}^\tau_s\tilde{\nabla} = (1 - s)^\phi\nabla + s\, {}^\tau\tilde{\nabla}$, $s \in [0, 1]$. Then the corresponding connection 1-form is ${}^\tau_s\tilde{\omega} = (1 - s)\omega + s\, {}^\tau\tilde{\omega}$ with curvature form ${}^\tau_s\tilde{\Omega}$. Now define the *transgression* as in [Gil95, Section 2.1 and Section 3.10.5] between ∇ and ${}^\tau\tilde{\nabla}$ for the $\hat{A}$-Genus by

$$T\hat{A}(\nabla, {}^\tau\tilde{\nabla}) := (\dim N) \int_0^1 \hat{A}({}^\tau\theta, \underbrace{{}^\tau_s\tilde{\Omega}, \dots, {}^\tau_s\tilde{\Omega}}_{\frac{1}{2}(\dim N - 1) \text{ times}}) \; \mathrm{ds} \in \Omega^{m-1}(M^\tau) \tag{45.1}$$

where ${}^\tau\theta := {}^\tau\tilde{\omega} - \omega$ is the difference between the connection 1-forms.

It is a fact that then

$$\mathrm{d}\, T\hat{A}(\nabla, {}^\tau\tilde{\nabla}) = \hat{A}(M, {}^\tau\tilde{g}) - \hat{A}(M, {}^\phi g) \tag{45.2}$$

(compare for example [Gil95, Lemma 2.1.2 (b) and Equation (2.1.40)]).

46. Lemma: Order of the Transgression

The transgression is of order -1 in τ so that

$$\lim_{\tau\to\infty} \int_{N^\tau} T\hat{A}(\nabla, {}^\tau\tilde{\nabla}) = 0. \tag{46.1}$$

Proof To do the necessary calculations we rely again on the product model M_∞ of the end and a local orthonormal ϕ-subordinate frame (e_i) of N^{t_0}, which has been extended through parallel continuation in T-direction to give $(\bar{e}_i)$ just like in Lemma 7. It is also there, that we gave the t-dependence of the Levi-Civita connection coefficients ${}^\phi g(\nabla_{\bar{e}_i}\bar{e}_j, \bar{e}_k)$ on N^t.

Ultimately we want to apply Definition 45 and compare the transgression $T\hat{A}(\nabla, {}^\tau\tilde{\nabla})\big|_{N^\tau}$ for different values of τ. To achieve that, we need to represent the forms ${}^\tau\theta^\tau$ and ${}^\tau_s\tilde{\Omega}^\tau$ in comparable terms, which are independent of τ. We will treat them as matrices of 1- and 2-forms with columns and rows defined by the local frame $(\bar{e}_i)$. Here the application of the characteristic polynomial $\hat{A}$ compensates for the fact that the $\bar{e}_i^H$ vary depending on t.

In contrast we must separate the τ dependence of the form-part of ${}^\tau\theta^\tau$ and ${}^\tau_s\tilde{\Omega}^\tau$ for the comparison to succeed. That means when starting from the values in Lemma 7 we get

$$\omega^t = \left(\begin{array}{c|ccc|ccc} 0 & -\frac{t_0}{t^2}\varepsilon_2^H & \cdots & -\frac{t_0}{t^2}\varepsilon_{\dim B+1}^H & 0 & \cdots & 0 \\ \hline \frac{t_0}{t^2}\varepsilon_2^H & & & & & & \\ \vdots & & \frac{t_0^2}{t^2}\omega_{HHH} + \frac{t_0^2}{t^2}\omega_{VHH} & & & \frac{t_0^3}{t^3}\omega_{HHV} + \frac{t_0}{t}\omega_{VHV} & \\ \frac{t_0}{t^2}\varepsilon_{\dim B+1}^H & & & & & & \\ \hline 0 & & & & & & \\ \vdots & & \frac{t_0^3}{t^3}\omega_{HVH} + \frac{t_0}{t}\omega_{VVH} & & & \frac{t_0^2}{t^2}\omega_{HVV} + \omega_{VVV} & \\ 0 & & & & & & \end{array}\right) \tag{46.2}$$

where $\varepsilon_i^H := g_\infty(e_i^H, \bullet) = \frac{t}{t_0} g_\infty(\bar{e}_i^H, \bullet)$ is the dual of e_i^H and ω_{***} are (suitably sized) matrices of 1-forms corresponding to $(\omega_{k,j}) = (\sum_i g_\infty(\nabla_{e_i^*}\bar{e}_j^*, \bar{e}_k^*)\varepsilon_i^*)_{k,j}$ for $t = t_0$ and in particular not depending on t or τ.

Define the following helper function for $\tau \in \mathbb{R}_+$ and $i \in \{1, \dots, \dim M\}$ on M_∞

$$\hbar(i,\tau)|_{N^t} := \begin{cases} \frac{\tau}{t} & \text{if } e_i \text{ is horizontal} \\ 1 & \text{otherwise} \end{cases} \tag{46.3}$$

then we know that $\bar{e}_i^H(t) = \frac{\tau}{t} e_i^H = \hbar(i,\tau)(t) e_i^H(\tau)$ as well as $\bar{e}_i^V(t) = e_i^V = \hbar(i,\tau)(t) e_i^V(\tau)$.

Furthermore denote by

$$\begin{aligned} \mathcal{K}(g,X,Y,Z) := {} & Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ & - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]) \end{aligned} \tag{46.4}$$

the Koszul formula, so that $g(\nabla_X^{LC} Y, Z) = \mathcal{K}(g,X,Y,Z)$.

Now we can calculate ${}^\tau\tilde{\omega}$ through

$${}^\tau\tilde{\nabla}_{e_i}\bar{e}_j(t) = {}^\tau\tilde{\nabla}_{e_i}\hbar(j,\tau)\cdot e_j(\tau) = (e_i\hbar(j,\tau))e_j(\tau) + \hbar(j,\tau)\cdot {}^\tau\tilde{\nabla}_{e_i}e_j(\tau) \tag{46.5}$$

$$\begin{aligned} & {}^\tau\tilde{g}^t({}^\tau\tilde{\nabla}_{e_i}\bar{e}_j(t), \bar{e}_k(t)) \\ & = \hbar(k,\tau)\cdot {}^\tau\tilde{g}^t((e_i\hbar(j,\tau))e_j(\tau) + \hbar(j,\tau)\cdot {}^\tau\tilde{\nabla}_{e_i}e_j(\tau), e_k(\tau)) \\ & = \hbar(k,\tau)\cdot(e_i\hbar(j,\tau))g^\tau(e_j(\tau),e_k(\tau)) + \hbar(k,\tau)\cdot\hbar(j,\tau)\cdot {}^\tau\tilde{g}^t({}^\tau\tilde{\nabla}_{e_i}\bar{e}_j(\tau),\bar{e}_k(\tau)) \\ & = \hbar(k,\tau)\cdot(e_i\hbar(j,\tau))\delta_{j,k} + \hbar(k,\tau)\cdot\hbar(j,\tau)\cdot\mathcal{K}({}^\tau\tilde{g}, e_i, \bar{e}_j(\tau), \bar{e}_k(\tau)) \end{aligned} \tag{46.6}$$

$$\begin{aligned} {}^\tau\tilde{\omega}^t_{k,j}(e_i) & = \frac{{}^\tau\tilde{g}^t({}^\tau\tilde{\nabla}_{e_i}\bar{e}_j(t),\bar{e}_k(t))}{{}^\tau\tilde{g}^t(\bar{e}_k,\bar{e}_k)} \\ & = \frac{\hbar(k,\tau)\cdot(e_i\hbar(j,\tau))\delta_{j,k} + \hbar(k,\tau)\cdot\hbar(j,\tau)\cdot\mathcal{K}({}^\tau\tilde{g}^t, e_i,\bar{e}_j(\tau),\bar{e}_k(\tau))}{\hbar(k,\tau)^2\cdot g^\tau(\bar{e}_k(\tau),\bar{e}_k(\tau))} \\ & = \frac{(e_i\hbar(j,\tau))\delta_{j,k} + \hbar(j,\tau)\cdot\mathcal{K}(g^\tau, e_i,\bar{e}_j(\tau),\bar{e}_k(\tau))}{\hbar(k,\tau)} \\ & = \frac{(e_i\hbar(j,\tau))\delta_{j,k}}{\hbar(k,\tau)} + \frac{\hbar(j,\tau)}{\hbar(k,\tau)}\omega^\tau_{k,j}(e_i) \end{aligned} \tag{46.7}$$

The first term of ${}^\tau\tilde{\omega}^t_{k,j}(e_i)$ vanishes, except for $j = k$ horizontal and $e_i = T$. In that case it is:

$$\frac{(e_i\hbar(j,\tau))\delta_{j,k}}{\hbar(k,\tau)} = \frac{t}{\tau}\frac{d}{dt}\frac{\tau}{t} = \frac{t}{\tau}\cdot\left(-\frac{\tau}{t^2}\right) = -\frac{1}{t} \tag{46.8}$$

We denote by I^H the unit matrix with $\dim B$ rows and columns. Than we can summarize

$$
{}^{\tau}\tilde{\omega}^{t} = \left(\begin{array}{c|c|c}
0 & -\frac{t_0^2}{\tau t}\varepsilon_2^H \quad \dots \quad -\frac{t_0^2}{\tau t}\varepsilon_{\dim B+1}^H & 0 \quad \cdots \quad 0 \\
\hline
\begin{matrix} \frac{t_0^2 t}{\tau^3}\varepsilon_2^H \\ \vdots \\ \frac{t_0^2 t}{\tau^3}\varepsilon_{\dim B+1}^H \end{matrix} & \frac{t_0^2}{\tau^2}\omega_{HHH} + \frac{t_0^2}{\tau^2}\omega_{VHH} - \frac{1}{t}\mathrm{d}t I^H & \frac{t_0^3}{\tau^2 t}\omega_{HHV} + \frac{t_0}{t}\omega_{VHV} \\
\hline
\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \frac{t_0^3 t}{\tau^4}\omega_{HVH} + \frac{t_0 t}{\tau^2}\omega_{VVH} & \frac{t_0^2}{\tau^2}\omega_{HVV} + \omega_{VVV}
\end{array}\right) \tag{46.9}
$$

Note that all terms in ω and ${}^{\tau}\tilde{\omega}$ are of the form $t^a \tau^b \alpha$ with $\alpha \in \Omega^1(TM|_{N^{t_0}})$, for which we can calculate

$$
\lim_{\tau\to\infty} (t^a \tau^b \alpha)\big|_{t=\tau} = \lim_{\tau\to\infty} \tau^{a+b}\alpha = \begin{cases} 0 & a+b<0 \\ \alpha & a+b=0 \\ \infty & a+b>0 \end{cases} \tag{46.10}
$$

$$
\begin{aligned}
\lim_{\tau\to\infty} (\mathrm{d}(t^a\tau^b\alpha))\big|_{t=\tau} &= \lim_{\tau\to\infty} (a t^{a-1}\tau^b \mathrm{d}t \wedge \alpha + t^a \tau^b \mathrm{d}\alpha)\big|_{t=\tau} \\
&= \lim_{\tau\to\infty} a\tau^{a+b-1}\mathrm{d}t\wedge\alpha + \tau^{a+b}\mathrm{d}\alpha = \begin{cases} 0 & a+b<0 \\ \mathrm{d}\alpha & a+b=0 \\ a\mathrm{d}t\wedge\alpha & a+b=1 \text{ and } \mathrm{d}\alpha=0 \\ \infty & \text{otherwise} \end{cases}
\end{aligned} \tag{46.11}
$$

and we further see that only $a+b \leq 0$ occurs. So the only term which does not vanish is the one where $a+b=0$, i.e. ω_{VVV}. Thus we find that

$$
\lim_{\tau\to\infty} \omega^\tau = \lim_{\tau\to\infty} {}^{\tau}\tilde{\omega}^{\tau} =: \omega^\infty \tag{46.12}
$$

$$
\lim_{\tau\to\infty} {}^{\tau}\theta^{\tau} = \lim_{\tau\to\infty} {}^{\tau}\tilde{\omega}^{\tau} - \omega^\tau = 0 \tag{46.13}
$$

$$
\begin{aligned}
{}^{\tau}_{s}\tilde{\Omega} &= \mathrm{d}\,{}^{\tau}_{s}\tilde{\omega} + ({}^{\tau}_{s}\tilde{\omega})^2 \\
&= (1-s)\mathrm{d}\omega + s\mathrm{d}\,{}^{\tau}\tilde{\omega} + ((1-s)\omega + s\,{}^{\tau}\tilde{\omega})^2
\end{aligned} \tag{46.14}
$$

$$
\begin{aligned}
\lim_{\tau\to\infty} {}^{\tau}_{s}\tilde{\Omega}^{\tau} &= (1-s)\mathrm{d}\omega^\infty + s\mathrm{d}\omega^\infty + ((1-s)\omega^\infty + s\omega^\infty) \\
&= \mathrm{d}\omega^\infty + \omega^\infty \wedge \omega^\infty =: \tilde{\Omega}^\infty
\end{aligned} \tag{46.15}
$$

All in all this gives us

$$\begin{aligned}
\lim_{\tau\to\infty}\int\limits_{N^\tau} T\hat{A}(\nabla, {}^\tau\tilde{\nabla}) &= \lim_{\tau\to\infty}\int\limits_{N^\tau}(m-1)\int\limits_0^1 \hat{A}({}^\tau\theta, \underbrace{{}^\tau_s\tilde{\Omega}, \dots, {}^\tau_s\tilde{\Omega}}_{\frac{1}{2}(\dim N - 1)\text{ times}})\ \mathrm{ds} \\
&= \lim_{\tau\to\infty}\int\limits_{N^\tau}(m-1)\int\limits_0^1 \hat{A}({}^\tau\theta^\tau, {}^\tau_s\tilde{\Omega}^\tau, \dots, {}^\tau_s\tilde{\Omega}^\tau)\ \mathrm{ds} \\
&= (m-1)\int\limits_N\int\limits_0^1 \hat{A}(\lim_{\tau\to\infty}{}^\tau\theta^\tau, \lim_{\tau\to\infty}{}^\tau_s\tilde{\Omega}^\tau, \dots, \lim_{\tau\to\infty}{}^\tau_s\tilde{\Omega}^\tau)\ \mathrm{ds} \\
&= (m-1)\int\limits_N\int\limits_0^1 \hat{A}(0, \tilde{\Omega}^\infty, \dots, \tilde{\Omega}^\infty)\ \mathrm{ds} \\
&= 0
\end{aligned} \tag{46.16}$$

exactly as claimed. ☑

47. Theorem: Adiabatic Limit (Bismut and Cheeger)

There is a form $\tilde{\eta}$ on the base B of the fibre bundle N such that

$$\lim_{\tau\to\infty}\eta(A^{\tau,+}) = \frac{1}{2}\lim_{\tau\to\infty}\eta(A^\tau) = -\int\limits_B \hat{A}(B, g_B)\wedge\tilde{\eta} \tag{47.1}$$

Proof Because we assumed that A^∞_V is invertible we can use [Dai91, Theorem 1.3] with $x = \frac{1}{\tau}$ and $D_x = -A^\tau$. Originally this result was published in [BC89, Equation (0.5)]. Our coefficient convention for $\tilde{\eta}$ is compatible with [LMP06]. The factor $\frac{1}{2}$ arises, because the η-invariant counts eigenvalues with their multiplicity and $A^\tau \simeq A^{\tau,+}\oplus(A^{\tau,+})^*$. Furthermore the sign change comes from the switch of orientation between x and t. ☑

Note that, as D is of Fredholm-type, the extended operator considered in [BBC12] is equal to the maximal extension $D_{\mathrm{ext}} = D_{\max}\colon H^1(M, S(M)) \to L^2(M, S(M))$. The restrictions of D^+ using spectral projections of A^τ, that we use below, are defined at the end of [BBC12, § 4.1].

48. Lemma: Vanishing Asymptotic Invariant

The asymptotic invariant $\mathrm{ind}\, D^+_{U^\tau,<-\lambda,\mathrm{ext}}$ vanishes for λ sufficiently small.

Proof According to [BBC12, Equation (5.20)] we know that

$$\mathrm{ind}\, D^+_{U^\tau,<-\lambda,\mathrm{ext}} = -\dim\ker D^+_{U^\tau,\le\lambda,\max}. \tag{48.1}$$

Now because A^τ is invertible it has a spectral gap around 0 and so the spectral projections corresponding to the intervals $(-\infty,-\lambda)$ and $(-\infty,\lambda]$ are equal for λ sufficiently small. This lets us conclude that

$$D^+_{U^\tau,\leq\lambda,\max} = D^+_{U^\tau,<-\lambda,\max} = D^+_{U^\tau,<-\lambda,\mathrm{ext}}. \tag{48.2}$$

Finally [BBC12, Proposition 4.46 (3)] says that $D_{<-\lambda,\mathrm{ext}}$ is injective and so is $D^+_{U^\tau,<-\lambda,\mathrm{ext}}$. Thus

$$\operatorname{ind} D^+_{U^\tau,<-\lambda,\mathrm{ext}} \overset{(48.1)}{=} -\dim\ker D^+_{U^\tau,\leq\lambda,\max} \overset{(48.2)}{=} -\dim\ker D^+_{U^\tau,<-\lambda,\mathrm{ext}} = 0, \tag{48.3}$$

which is what we wanted to show. ☑

49. Theorem: Index Formula

Now together we have

$$\operatorname{ind} D^+_{\mathrm{ext}} = \int_M \hat{A}(M,{}^\phi g) - \tfrac{1}{2}\int_B \hat{A}(B,g_B)\wedge\tilde{\eta} \tag{49.1}$$

Proof According to [BBC12, Theorem 5.13] we have

$$\operatorname{ind} D^+_{\mathrm{ext}} = \operatorname{ind} D^+_{M^\tau,\geq} + \dim H^+_{[-\lambda,0)} + \operatorname{ind} D^+_{U^\tau,<-\lambda,\mathrm{ext}}. \tag{49.2}$$

Applying [BBC12, Equation (5.21)] we get

$$\begin{aligned}\operatorname{ind} D^+_{\mathrm{ext}} = \int_{M^\tau} \hat{A}(M,{}^\phi g) + \int_{N^\tau} T\hat{A}(\nabla,\tilde{\nabla}^\tau) + \tfrac{1}{2}(\eta(A^{\tau,+}) + \dim\ker A^{\tau,+})\\ + \dim H^+_{[-\lambda,0)} + \operatorname{ind} D^+_{U^\tau,<-\lambda,\mathrm{ext}}\end{aligned} \tag{49.3}$$

(compare also [APS74, Theorem (4·2)]).

In Corollary 43 we showed that $\dim\ker A^{\tau,+}$ vanishes and because of the accompanying spectral gap $\dim H^+_{[-\lambda,0)}$ also vanishes for λ sufficiently small. Furthermore $\operatorname{ind} D^+_{U^\tau,<-\lambda,\mathrm{ext}}$ vanishes for λ sufficiently small according to Lemma 48.

Taking the limit for $\tau\to\infty$ we can apply Lemma 46 and Theorem 47, so that

$$\begin{aligned}\operatorname{ind} D^+_{\mathrm{ext}} &= \lim_{\tau\to\infty}\int_{M^\tau} \hat{A}(M,{}^\phi g) + \int_{N^\tau} T\hat{A}(\nabla,\tilde{\nabla}^\tau) + \tfrac{1}{2}\eta(A^{\tau,+})\\ &= \int_M \hat{A}(M,{}^\phi g) - \tfrac{1}{2}\int_B \hat{A}(B,g_B)\wedge\tilde{\eta}\end{aligned} \tag{49.4}$$

just as we wanted to show. ☑

A. Horizontal Coefficients of the Anti-Commutator $\tilde{D}_{HV}$

In [Brü09, Theorem 1.2] the author states that the modified anti-commutator $\tilde{D}_{HV}$ is first-order vertical. As the calculation given there is not quite detailed, we repeat it here. We rely heavily on the basic properties of a Dirac bundle as stated in Definition 11.

First repeat some definitions:

$$\begin{aligned}
\tilde{D}_H &= \sum_{i \text{ hor.}} c_i^H \nabla_{e_i^H} & \tilde{D}_V &= \sum_{i \text{ vert.}} c_i^V \nabla_{e_i^V} \\
\beta_1 &= -\dim \mathcal{F}\, \mathrm{Cl}(H_{\mathcal{F}}) & H_{\mathcal{F}} &= -\frac{1}{\dim \mathcal{F}} \sum_{i \text{ vert.}} \phi_{T^H N} \nabla_{e_i} e_i \\
D_H &= \tilde{D}_H - \tfrac{1}{2}\beta_1 & D_V &= \tilde{D}_V + \tfrac{1}{2}\beta_1 \\
\tilde{D}_{HV} &= \left\{ D_H + \beta_2, D_V - \beta_2 \right\} & \beta_2 &= \tfrac{1}{4} \sum_{\substack{i,k \text{ hor.} \\ j \text{ vert.}}} g(\nabla_{e_i^H} e_k^H, e_j^V) c_j^V c_k^H c_i^H
\end{aligned} \tag{A-1}$$

According to [Brü09, Lemma 1.1.2] we know that for a horizontal vector field X we have the relation $\left\{ \mathrm{Cl}(X), D_V \right\} = 0$. Now we start by calculating $\left\{ D_H, D_V \right\}$. Let R denote the Riemannian curvature tensor.

$$\begin{aligned}
\tilde{D}_H \tilde{D}_V &= \sum_{\substack{i \text{ hor.} \\ j \text{ vert.}}} c_i^H \nabla_{e_i^H} c_j^V \nabla_{e_j^V} = \sum_{\substack{i \text{ hor.} \\ j \text{ vert.}}} \left(c_i^H c_j^V \nabla_{e_i^H} \nabla_{e_j^V} + c_i^H \,\mathrm{Cl}\left(\nabla_{e_i^H} e_j^V \right) \nabla_{e_j^V} \right) \\
&= \sum_{\substack{i \text{ hor.} \\ j \text{ vert.}}} \left(c_i^H \,\mathrm{Cl}\left(\nabla_{e_i^H} e_j^V \right) \nabla_{e_j^V} - c_j^V c_i^H \nabla_{e_j^V} \nabla_{e_i^H} - c_j^V c_i^H R(e_i^H, e_j^V) - c_j^V c_i^H \nabla_{[e_j^V, e_i^H]} \right)
\end{aligned} \tag{A-2}$$

$$\tilde{D}_V \tilde{D}_H = \sum_{\substack{i \text{ hor.} \\ j \text{ vert.}}} c_j^V \nabla_{e_j^V} c_i^H \nabla_{e_i^H} = \sum_{\substack{i \text{ hor.} \\ j \text{ vert.}}} \left(c_j^V c_i^H \nabla_{e_j^V} \nabla_{e_i^H} + c_j^V \,\mathrm{Cl}\left(\nabla_{e_j^V} e_i^H \right) \nabla_{e_i^H} \right) \tag{A-3}$$

$$\begin{aligned}
\left\{ \tilde{D}_H, \tilde{D}_V \right\} = \sum_{\substack{i \text{ hor.} \\ j \text{ vert.}}} \Bigg(& \underbrace{c_i^H \,\mathrm{Cl}\left(\nabla_{e_i^H} e_j^V \right) \nabla_{e_j^V}}_{\text{vertical}} + c_j^V \,\mathrm{Cl}\left(\nabla_{e_j^V} e_i^H \right) \nabla_{e_i^H} \\
& \underbrace{- c_j^V c_i^H \nabla_{[e_j^V, e_i^H]}}_{\text{vertical}} \underbrace{- c_j^V c_i^H R(e_i^H, e_j^V)}_{\text{0th order}} \Bigg)
\end{aligned} \tag{A-4}$$

$$\left\{\tilde{D}_H, \beta_1\right\} = \left\{\sum_{i \text{ hor.}} c_i^H \nabla_{e_i^H}, -\dim \mathcal{F} \cdot \mathrm{Cl}(H_{\mathcal{F}})\right\}$$
$$= -\dim \mathcal{F} \cdot \sum_{i \text{ hor.}} \left(c_i^H \nabla_{e_i^H} \mathrm{Cl}(H_{\mathcal{F}}) + \mathrm{Cl}(H_{\mathcal{F}}) c_i^H \nabla_{e_i^H} \right)$$
$$= -\dim \mathcal{F} \cdot \sum_{i \text{ hor.}} \Big(\underbrace{c_i^H \mathrm{Cl}\left(\nabla_{e_i^H} H_{\mathcal{F}}\right)}_{\text{0th order}} + c_i^H \mathrm{Cl}(H_{\mathcal{F}}) \nabla_{e_i^H} + Cl(H_{\mathcal{F}}) c_i^H \nabla_{e_i^H} \Big) \tag{A-5}$$

$$\left\{D_H, D_V\right\} = \left\{\tilde{D}_H - \tfrac{1}{2}\beta_1, \tilde{D}_V + \tfrac{1}{2}\beta_1\right\} = \left\{\tilde{D}_H, \tilde{D}_V\right\} + \tfrac{1}{2}\left\{\tilde{D}_H, \beta_1\right\} - \underbrace{\tfrac{1}{2}\left\{\beta_1, D_V\right\}}_{\substack{= 0, \text{ because} \\ H_{\mathcal{F}} \text{ is horizontal}}} \tag{A-6}$$

So we can also calculate the horizontal part of $\tilde{D}_{HV}$, whose coefficients are denoted by γ_n.

$$\left\{D_H, \beta_2\right\} = \frac{1}{4} \sum_{\substack{i,k,l \text{ hor.} \\ j \text{ vert.}}} \left\{ c_l^H \nabla_{e_l^H}, g(\nabla_{e_i^H} e_k^H, e_j^V) c_j^V c_k^H c_i^H \right\}$$
$$= \frac{1}{4} \sum_{\substack{i,k,l \text{ hor.} \\ j \text{ vert.}}} \Big(\underbrace{c_l^H e_l^H (g(\nabla_{e_i^H} e_k^H, e_j^V)) c_j^V c_k^H c_i^H + c_l^H g(\nabla_{e_i^H} e_k^H, e_j^V) \mathrm{Cl}\left(\nabla_{e_l^H} e_j^V\right) c_k^H c_i^H}_{\text{0th order}}$$
$$+ \underbrace{c_l^H g(\nabla_{e_i^H} e_k^H, e_j^V) c_j^V \mathrm{Cl}\left(\nabla_{e_l^H} e_k^H\right) c_i^H + c_l^H g(\nabla_{e_i^H} e_k^H, e_j^V) c_j^V c_k^H \mathrm{Cl}\left(\nabla_{e_l^H} e_i^H\right)}_{\text{0th order}}$$
$$+ \underbrace{c_l^H g(\nabla_{e_i^H} e_k^H, e_j^V) c_j^V c_k^H c_i^H \nabla_{e_l^H} + g(\nabla_{e_i^H} e_k^H, e_j^V) c_j^V c_k^H c_i^H c_l^H \nabla_{e_l^H}}_{\text{sums up over } i,j,k \text{ to give } 4\left\{c_l, \beta_2\right\} \nabla_{e_l^H}} \Big) \tag{A-7}$$

$$\tilde{D}_{HV} = \left\{D_H, D_V\right\} + \underbrace{\left\{\beta_2, D_V\right\}}_{\text{vertical}} - \left\{D_H, \beta_2\right\} - \underbrace{\left\{\beta_2, \beta_2\right\}}_{\text{0th order}}$$
$$= \left\{\tilde{D}_H, \tilde{D}_V\right\} + \tfrac{1}{2}\left\{\tilde{D}_H, \beta_1\right\} - \sum_{i \text{ hor.}} \left\{c_i^H, \beta_2\right\} \nabla_{e_i^H} + \overset{\text{vertical}}{\cdots}$$
$$= \sum_{\substack{i \text{ hor.} \\ j \text{ vert.}}} c_j^V \mathrm{Cl}\left(\nabla_{e_j^V} e_i^H\right) \nabla_{e_i^H} - \tfrac{1}{2} \dim \mathcal{F} \cdot \sum_{i \text{ hor.}} \left\{c_i^H, \mathrm{Cl}(H_{\mathcal{F}})\right\} \nabla_{e_i^H}$$
$$- \sum_{i \text{ hor.}} \left\{c_i^H, \beta_2\right\} \nabla_{e_i^H} + \overset{\text{vertical}}{\cdots} \tag{A-8}$$

$$\begin{aligned}
\left\{c_n^H, \beta_2\right\} &= \frac{1}{4} \sum_{\substack{i,k \text{ hor.} \\ j \text{ vert.}}} g(\nabla_{e_i^H} e_k^H, e_j^V) \left\{ c_j^V c_k^H c_i^H, c_n^H \right\} \\
&= \frac{1}{4} \sum_{\substack{i \text{ hor.} \\ j \text{ vert.}}} \left(g(\nabla_{e_i^H} e_n^H, e_j^V) \left\{ c_j^V c_n^H c_i^H, c_n^H \right\} + g(\nabla_{e_n^H} e_i^H, e_j^V) \left\{ c_j^V c_i^H c_n^H, c_n^H \right\} \right) \\
&= \frac{1}{2} \sum_{\substack{i \text{ hor.} \\ j \text{ vert.}}} \left(g(\nabla_{e_i^H} e_n^H, e_j^V) c_j^V c_i^H - g(\nabla_{e_n^H} e_i^H, e_j^V) c_j^V c_i^H \right) \\
&= \frac{1}{2} \sum_{\substack{i \text{ hor.} \\ j \text{ vert.}}} g([e_i^H, e_n^H], e_j^V) c_j^V c_i^H
\end{aligned} \tag{A-9}$$

$$\frac{1}{2} \dim \mathcal{F} \cdot \left\{ c_n^H, \mathrm{Cl}(H_{\mathcal{F}}) \right\} = -\dim \mathcal{F} \cdot g(e_n^H, H_{\mathcal{F}}) = \sum_{j \text{ vert.}} g(e_n^H, \nabla_{e_j^V} e_j^V) \tag{A-10}$$

$$\begin{aligned}
\gamma_n &= \sum_{j \text{ vert.}} c_j^V \,\mathrm{Cl}\left(\nabla_{e_j^V} e_n^H \right) - \frac{1}{2} \dim \mathcal{F} \cdot \left\{ c_n^H, \mathrm{Cl}(H_{\mathcal{F}}) \right\} - \left\{ c_n^H, \beta_2 \right\} \\
&= \sum_{\substack{i \text{ hor.} \\ j \text{ vert.}}} g(\nabla_{e_j^V} e_n^H, e_i^H) c_j^V c_i^H + \cancel{\sum_{j,k \text{ vert.}} g(\nabla_{e_j^V} e_n^H, e_k^V) c_j^V c_k^V} \\
&\quad - \cancel{\sum_{j \text{ vert.}} g(e_n^H, \nabla_{e_j^V} e_j^V)} - \frac{1}{2} \sum_{\substack{i \text{ hor.} \\ j \text{ vert.}}} g([e_i^H, e_n^H], e_j^V) c_j^V c_i^H \\
&= -\frac{1}{2} \sum_{\substack{i \text{ hor.} \\ j \text{ vert.}}} g(e_j^V, [e_n^H, e_i^H]) c_j^V c_i^H - \frac{1}{2} \sum_{\substack{i \text{ hor.} \\ j \text{ vert.}}} g([e_i^H, e_n^H], e_j^V) c_j^V c_i^H \\
&= 0
\end{aligned} \tag{A-11}$$

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