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Peetre's Theorem

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Contents

XeTeX The code was written in `gvim` with the help of `vim-latex` and compiled with `xelatex`, all on `Gentoo Linux`. Thanks to the free software community and to the fellow TeX users on TeX.SX for their great help and advice.
If you want to make any comments or suggest corrections to this script, please contact me! You can either email me directly or use the contact form on my website.

In 1959 JAAK PEETRE first published his astonishing result, that any local (i.e. support-non-increasing) operator is a differential operator, [Pee59]. Unfortunately his proof contained an error and so in 1960 he had to publish a correction, [Pee60]. There, instead of just patching the error of the previous paper, he proposed a more general version of his theorem using distribution theory. We will take this second approach as it seems more natural. So before we can proof PEETRE's Theorem we need to introduce the necessary results of distribution theory. Most of those are taken from [Hö90]. For a more abstract context [RR73] is a good source.

1 Basics

Choose any open set $\Omega \subset \mathbb{R}^n$ and a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

1 **Definition: "Test functions"**

Let $U \subset \Omega$, then we define

$$\mathcal{D}(U) := \{ f \in C^\infty(\Omega, \mathbb{K}) \mid U \supset \operatorname{supp} f \text{ compact} \}$$

and call these functions *test functions* with support in U.

For compact $U = K$, we take on $\mathcal{D}(K)$ the topology defined by the semi-normes:

$$\|f\|_{K,m} := \sup_{\substack{|\alpha| \leq m \\ x \in K}} |D^\alpha f(x)|$$

for $m \in \mathbb{N}$ and α denoting a multi-index. Then $\|f\|_{K,m} \leq \|f\|_{K,m+1}$.

Now we choose the topology on $\mathcal{D}(\Omega)$ as the inductive limit under the inclusions $\mathcal{D}(K) \to \mathcal{D}(\Omega)$, $K \subset \Omega$ compact. It is the coarsest topology which induces a finer topology on $\mathcal{D}(K)$ than the given one or in other words the topology under which an operator is continuous iff its restriction to every $\mathcal{D}(K)$ is continuous.

2 **Definition: "Distribution"**

The continuous dual space of $\mathcal{D}(\Omega)$ under this topology is called the space of *distributions* on Ω and denoted by $\mathcal{D}'(\Omega)$.

On that space in turn, we put the weak*-topology, i.e. the topology induced by the semi-norms

$$\|F\|_f := |F(f)| \qquad f \in \mathcal{D}(\Omega)$$

On every $\mathcal{D}'(U)$ we can also consider the (not necessarily finite) operator norms

$$\|F\|'_{U,m} := \sup_{f \in \mathcal{D}(U)} \frac{|F(f)|}{\|f\|_m}$$

Then we have $\|F\|'_{U,m} \geq \|F\|'_{U,m+1}$ and

$$\forall \text{compact } K, F \in \mathcal{D}'(K) \exists m_K \in \mathbb{N} \colon \|F\|'_{K,m_K} < \infty \tag{1-1}$$

This (smallest possible) m is called the order of the distribution F on K.

3 **Definition: "Tensor Product"**

Let $U_i \subset \mathbb{R}^{n_i}$, $i = 1, 2$ open, then we define: For any $f_i \in C(U_i)$ their *tensor product* $f_1 \otimes f_2 \colon C(U_1 \times U_2)$ is

$$(f_1 \otimes f_2)(x_1, x_2) := f_1(x_1) f_2(x_2)$$

4 **Theorem:** SCHWARTZ **Kernel Theorem**

The equation

$$K(g)(f) = k(f \otimes g), \qquad f \in \mathcal{D}(U_1), g \in \mathcal{D}(U_2) \tag{4.1}$$

defines a bijective relation between distributions $k \in \mathcal{D}'(U_1 \times U_2)$ and continuous linear maps $K \colon \mathcal{D}(U_2) \to \mathcal{D}'(U_1)$.

Compare [Hö90, 5.2.1, 128ff]

5 **Remark:**

What we are actually interested in, are the distributions with compact support. They can in a natural way be seen as the continuous functionals on the space $\mathcal{E}(\Omega)$. This space is $C^\infty(\Omega)$ with the topology given by the semi-norms $\|\cdot\|_{K,m}$, $K \subset \Omega$ compact and $m \in \mathbb{N}$. Notice that the induced topology on $\mathcal{D}(\Omega)$ as a topological subspace of $\mathcal{E}(\Omega)$ is coarser than the one above. Thus $\mathcal{D}'(\Omega) \supset \left\{ F \in \mathcal{D}'(\Omega) \;\middle|\; \operatorname{supp} F \text{ compact} \right\} \simeq \mathcal{E}'(\Omega)$. The extension of a distribution $F \in \mathcal{D}'(\Omega)$ with compact support to functions without compact support can be done by using appropriate cut-off functions.

6 **Theorem:**

Let $F \in \mathcal{E}'(\Omega)$ with order at most m and $\phi \in C^{m+1}(\Omega)$ such that

$$\forall x \in \operatorname{supp} F, |\alpha| \le m \colon D^\alpha \phi(x) = 0$$

then already $F(\phi) = 0$, i.e. a distribution with compact support and order m only depends on the derivatives up to order m.

Compare [Hö90, 2.3.3, 46]

Proof (6) We accept at this point, that there are cut-off functions $\chi_\varepsilon \in \mathcal{D}(\Omega)$ with $\chi_\varepsilon \equiv 1$ on a open neighbourhood of $\operatorname{supp} F$ and $\chi_\varepsilon \equiv 0$ on the complement of

$$M_\varepsilon = \operatorname{supp} u + B_\varepsilon$$

denoting the standard ε-Ball by B_ε and $+$ for element-wise addition. These χ_ε can also be chosen in such a way, that

$$\forall \alpha, |\alpha| \le m \colon |D^\alpha \chi_\varepsilon| \le C \varepsilon^{-|\alpha|}$$

holds. (See [Hö90, 1.4.1, 25])

Now on $\operatorname{supp} F$ we have $\phi = \phi \chi_\varepsilon$ and so $u(\phi) = u(\phi \chi_\varepsilon)$. Thus the following estimate holds:

$$|F(\phi)| \le C_1 \sum_{|\alpha| \le m} \sup |D^\alpha (\phi \chi_\varepsilon)| \le C_2 \sum_{|\alpha|+|\beta| \le m} \sup |D^\alpha \phi| |D^\beta \chi_\varepsilon| \le C_3 \sum_{|\alpha| \le m} \varepsilon^{|\alpha|-m} \sup_{M_\varepsilon} |D^\alpha \phi|$$

To complete the proof we will show that under the given preconditions

$$\forall\alpha, |\alpha| \leq m \colon \lim_{\varepsilon\to 0} \varepsilon^{|\alpha|-m} \sup_{M_\varepsilon} |D^\alpha\phi| = 0$$

For $|\alpha| = m$ this follows because $\forall x \in M_\varepsilon \colon \mathrm{dist}(x, \mathrm{supp}\, F) \leq \varepsilon$, $D^\alpha\phi$ is uniformly continuous and vanishes on supp F.

For $|\alpha| < m$, $y \in M_\varepsilon$ and $x \in \mathrm{supp}\, F$ s.t. $|x - y| \leq \varepsilon$ Taylor's formula gives us:

$$|D^\alpha\phi(y)| \leq \frac{1}{(m-|\alpha|)!} \sup_{0<t<1} |(\frac{d}{dt})^{m-|\alpha|}(D^\alpha\phi)(x + t(y-x))|$$

as all derivatives of order lower than m of ϕ vanish at $y = x$.

According to the chain rule the differentiation wrt t yields a sum of products of $(y-x)^{m-|\alpha|}$ and some m-th order derivative of ϕ. So because $|x - y| \leq \varepsilon$ and the derivatives of order m of ϕ are continuous and vanish on supp F, so the claim is proven for any α.

7 **Theorem: Splitting variables**

Regard $\mathbb{R}^n$ as $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and thus $x = (x_1, x_2)$. If now the support of a distribution $F \in \mathcal{D}'(\Omega)$ of order m is contained in the subspace $\{0\} \times \mathbb{R}^{n_2}$, F has the form

$$F(\phi) = \sum_{|\alpha|\leq m} F_\alpha(\phi_\alpha)$$

with certain distributions F_α in x_2 having compact support and

$$\phi_\alpha(x_2) = D^\alpha_{x_1}\phi(x_1, x_2)\big|_{x_1=0}$$

Notice that this can be done using any linear subspace and some complement.

Compare [Hö90, 2.3.5, 47f]

Proof (7) Let $\phi \in \mathcal{D}(\Omega)$, then we can compute its Taylor expansion in x_1 as

$$\phi(x) = \sum_{\substack{|\alpha|\leq m \\ \alpha_2=0}} D^\alpha\phi(0, x_2)\frac{x_1^\alpha}{\alpha!} + r(x)$$

where the multi-index is split as $\alpha = (\alpha_1, \alpha_2)$ just like the space itself.

Here $D^\alpha r(x) = 0$ when $x_1 = 0$ as long as $|\alpha| \leq k$. So the conditions of Theorem 6 are fulfilled and so $F(r) = 0$. Also we have

$$F_\alpha(\phi) = F(\phi(x_2)\frac{x_1^\alpha}{\alpha!})$$

8 **Theorem: Diagonal support**

The kernel k of a continuous map $K: \mathcal{D}(U) \to \mathcal{D}'(U)$ is supported in the diagonal iff it has the form

$$K(f) = \sum_\alpha a_\alpha D^\alpha f$$

for $a_\alpha \in \mathcal{D}'(U)$ and this sum is locally finite.

Compare [Hö90, 5.2.3, 131]

Proof (8) If K has the given form, the kernel is

$$k(\phi) = \sum_\alpha a_\alpha \; D_y^\alpha(\phi(x,y))\big|_{x=y}$$

because it meets Equation 4.1. This kernel is certainly supported in the diagonal.

If on the other hand K is given with a kernel k which is supported in the diagonal, we can apply Theorem 7 and get the above form for the kernel with a locally finite sum.

2 Peetre's Theorem

Let $P \in L(\mathcal{D}(\Omega), \mathcal{D}'(\Omega))$ not necessarily continuous, with the *locality property*:

$$\operatorname{supp} Pf \subset \operatorname{supp} f \qquad \forall f \in \mathcal{D}(\Omega) \tag{2-1}$$

Define also:

$$j(m,x) := \inf \left\{ j \in \mathbb{N} \;\middle|\; \exists U \ni x \colon \sup_{f \in \mathcal{D}(U)} \frac{\|Pf\|'_{U,m}}{\|f\|_j} < \infty \right\}$$

9 **Definition: "Point of Continuity"**

We say that $x \in \Omega$ is a *point of continuity* of P iff there is a neighbourhood $U \ni x$ such that $P|_{\mathcal{D}(U)}$ is continuous. Otherwise it is a *point of discontinuity*. The set of points of discontinuity let us denote by Λ. Then for $x \in \Lambda$ and all m we have $j = \infty$ and for $x \notin \Lambda$ there is a m such that $j < \infty$.

10 **Lemma:**

Λ is a discrete set and the function $j(m,\cdot)$ is locally bounded on $\Omega \setminus \Lambda$ for m sufficiently big.

Proof (10) It is enough to show that there is no sequence of pairwise different points $(x_i)_{i\in\mathbb{N}}$, $x_i \in K$ compact and sequences $(j_i), (m_i)$ diverging to $+\infty$ with $j_i < j(m_i, x_i)$. Assume such sequences are given, then we can can find disjoint $K_i \ni x_i$ and functions $f_i \in \mathcal{D}(K_i)$ with

$$\|f_i\|_{j_i} = 2^{-i} \quad \wedge \quad \|Pf_i\|'_{K,m_i} \geq 2^i$$

Now define $f(x) = f_i(x)$ for $x \in K_i$ and $f(x) = 0$ otherwise. Then $Pf = Pf_i$ in K_i and so we get

$$\|Pf\|'_{K,m_i} \geq \|Pf_i\|'_{K_i,m_i} \geq 2^i$$

for every i which is contradiction to Equation 1-1

11 **Theorem:** (PEETRE)

For any such local P as defined above there is a locally finitely supported family of distributions $\{a^\alpha\}$, unique on $\Omega \setminus \Lambda$, such that

$$\forall f \in \mathcal{D}(\Omega)\colon \operatorname{supp}(Pf - \sum_\alpha a^\alpha D^\alpha f) \subset \Lambda$$

Proof (11) According to Lemma 10 there are $j, m \in \mathbb{N}$ such that for any $x_0 \in \Omega \setminus \Lambda$ there is a neighbourhood U of x_0 with

$$|Pf(g)| \leq C\|f\|_j\|g\|_m$$

For all $f, g \in \mathcal{D}(U)$ and some fixed constant C. Thus SCHWARTZ Kernel Theorem says, that there is a distribution $p \in \mathcal{D}'(U \times U)$ (called the kernel of P) with

$$Pf(g) = p(f \otimes g)$$

Because of Equation 2-1 p is supported in the diagonal D of $U \times U$. In consequence we can apply Theorem 8: For all $h \in \mathcal{D}(U \times U)$ we have

$$Pf = \sum_\alpha a_\alpha D^\alpha f$$

with a_α distributions in $\mathcal{D}'(U)$ and α multi-indices up to a certain order.

By considering a slightly smaller set $\tilde{U} \subset U$ and choosing f constant to 1 on $\tilde{U}$, we get $a_0 = Pf$. Similar formulas hold for all a_α when choosing the monomials x^α. Thus all a_α are uniquely determined only by P.

Finally the formula can be extended from the $\tilde{U}$ to all of $\Omega \setminus \Lambda$ by choosing a locally finite refinement and taking a subordinate partition of unity.

12 **Corollary:**

If $H \subset \mathcal{D}'(\Omega)$ is a subspace which is closed under multiplication with $\mathcal{D}(\Omega)$-functions and $\operatorname{im} P \subset H$, than the a_α can be chosen to lie in H.

In particular H can be chosen to be the embedding of $\mathcal{D}(\Omega)$ in $\mathcal{D}'(\Omega)$ and the theorem then says, that every local morphism of the sheaf $\mathcal{D}(\Omega)$ is a differential operator.

13 **Remark:**

The theorem can be applied to local linear operators between sections of vector bundles by choosing local trivializations.

Further generalisations of the case $H = \mathcal{D}(\Omega)$ for certain Banach spaces and non-linear operators can be found in [WD73] and [KMS93, V.19, 176] respectively.

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