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Sommersemester 2012

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April 3, 2014

XƎTEX The code was written in gvim with the help of vim-latex and compiled with xelatex, all on Gentoo Linux. Thanks to the free software community and to the fellow TEX users on TEX.SX for their great help and advice.
If you want to make any comments or suggest corrections to this script, please contact me! You can either email me directly or use the contact form on my website.

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I. The analytic index of elliptic operators: Abstract operator theory

I.1. Fredholm Operators

In the following we will denote by H and $H_i, i \in \mathbb{N}$ seperable complex Hilbert spaces with scalar product $\langle \cdot, \cdot \rangle_{H_i}$. But usually we write just $\langle \cdot, \cdot \rangle$ when the space is obvious from the context. The corresponding norm is written as $\| \cdot \|_{H_i}$.

1. **Definition: "Closed Operator"**

A *linear operator* $A: H_1 \supset \operatorname{dom} A \to \operatorname{im} \subset H_2$ is called *closed* iff its graph

$$\operatorname{gr} A := \{ (v, Av) \mid v \in \operatorname{dom} A \} \subset H_1 \times H_2 \tag{1.1}$$

is closed. The set of *densely defined*, *closed operators* will be denoted by

$$C(H_1, H_2) := \left\{ A: \operatorname{dom} A \to H_2 \;\middle|\; \overline{\operatorname{dom} A} = H_1, \operatorname{gr} A \text{ closed} \right\} \tag{1.2}$$

From now on we will always mean densely defined, closed operator when we say *closed operator*.

2. **Remark:**

- If $\operatorname{dom} A = H_1$, A is closed if and only if A is continuous. In other words this means that $C(H_1, H_2)$ contains $\mathscr{L}(H_1, H_2)$.
- For $A \in C(H_1, H_2)$, although $\ker A \subset H_1$ is always closed, $\operatorname{im} A \subset H_2$ need not be closed.
- The *cokernel* is:

$$\operatorname{coker} A := H_2 / \operatorname{im} A \tag{2.1}$$

- If A is closed, so is its adjoint: $A^* \in C(H_2, H_1)$; because $\operatorname{gr} A^*$ is a rotation of $(\operatorname{gr} A)^\perp$.

3. **Lemma: Finite Cokernel**

P If the cokernel of a closed operator is finite dimensional, its image is closed.

4. **Definition: "Fredholm Operator"**

A closed operator is called *Fredholm* iff it has finite dimensional kernel and cokernel. We denote the set of *Fredholm operators* by

$$\mathscr{F}(H_1, H_2) := \left\{ A \in C(H_1, H_2) \;\middle|\; \dim \ker A + \dim \operatorname{coker} A < \infty \right\} \tag{4.1}$$

For $A \in \mathscr{F}(H_1, H_2)$ its *index* is defined as

$$\operatorname{ind} A := \dim \ker A - \dim \operatorname{coker} A = \dim \ker A - \dim \ker A^* \in \mathbb{Z} \tag{4.2}$$

For bounded Fredholm operators we write $\mathscr{F}_{bd}(H_1, H_2) := \mathscr{F}(H_1, H_2) \cap \mathscr{L}(H_1, H_2)$.

Note: If $A \in \mathscr{F}(H_1, H_2)$ Lemma 3 (Finite Cokernel) tells us that im A is closed.

5. **Definition: "Semi-Fredholm Operator"**

If $A \in C(H_1, H_2)$ only satisfies the weaker assumption, that ker A is finite dimensional and im A is closed, we call it *semi-Fredholm*.

6. **Lemma: Semi-Fredholm Condition**

$A \in C(H_1, H_2)$ is *semi-Fredholm* iff every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in dom A with (Ax_n) convergent in H_2 has a convergent subsequence.

Proof (6)

"⇒" $(x_n) \subset H_1$, $\|x_n\| \leq 1$, $(Ax_n) \to y$ in H_2.

Decompose $x_n = x_n^{\ker} + x_n^{\text{coker}}$, $x_n^{\ker} \in \ker A$, $\|x_n^{\ker}\|^2 + \|x_n^{\text{coker}}\|^2 \leq 1$, $x_n^{\text{coker}} \perp x_n^{\ker}$. Since ker A is finite dimensional, we may assume that $(x_n^{\ker}) \to x^{\ker}$. Thus we may assume that $\forall n\colon x_n \perp \ker A$.

Now we claim that, for some $\delta > 0$

$$\|Ax_n\| \geq \delta \|x_n\| \tag{6.1}$$

We take $\tilde{A} := A|_{(\ker A)^\perp}$, then $\tilde{A}$ is a bijective $(\ker A)^\perp \to \operatorname{im} A$ and because im A is closed, $\tilde{A}^{-1}$ is continuous by the open mapping theorem. Thus

$$\|x\| = \|\tilde{A}^{-1}\tilde{A}x\| \leq C\|\tilde{A}x\| = C\|Ax\| \forall x \perp \ker A \tag{6.2}$$

"⇐" $(x_n) \subset \ker A$, $\|x_n\| \leq 1$ then $(Ax_n) = (0)$ is convergent thus (x_n) has a convergent subsequence ⇒ $B(\ker A)$ is compact ⇒ $\dim \ker A < \infty$.

So let $(x_n) \perp \ker A$ s.t. $y_n = Ax_n \to y$. If we show that $\|Ax_n\| \geq \delta \|x_n\|$ for some $\delta > 0$, then we are done. Assume not, i.e. $\|Ax_n\| \leq \frac{1}{n}\|x_n\|$, $\|x_n\| = 1$ then $Ax_n \to 0$ and wlog $x_n \to x \Rightarrow \|x\| = 1 \Rightarrow Ax_n \to Ax = 0$. This is impossible since $x \perp \ker A$.

7. **Definition: "Compact Operator"**

If the image of the unit sphere in H_1 under $A \in \mathscr{L}(H_1, H_2)$ has compact closure in H_2, we call A a *compact operator*. That means we require the image of every bounded sequence in H_1 under A to contain a Cauchy subsequence. (This property automatically implies boundedness.) The space of all compact operators is denoted by $\mathcal{K}(H_1, H_2)$.

8. **Lemma: Properties of Fredholm Operators**

Let $A \in \mathscr{F}(H_1, H_2)$ then:

1. $A^* \in \mathscr{F}(H_2, H_1)$
2. There is an orthogonal decomposition

$$H_1 = \ker A \oplus \operatorname{im} A^*, \qquad H_2 = \ker A^* \oplus \operatorname{im} A \tag{8.1}$$

3. There is $R_0 \in \mathscr{L}(H_1, H_1)$ such that

$$\mathrm{id}_{H_1} - R_0 A = P_{\ker A} \in \mathcal{K}(H_1) \text{and} \tag{8.2}$$
$$\mathrm{id}_{H_2} - A R_0 = P_{\ker A^*} \in \mathcal{K}(H_2). \tag{8.3}$$

This implies FREDHOLM's Theorem:

$$\mathrm{id}_{H_1} - P_{\ker A} \in \mathscr{F}(H_1), \mathrm{id}_{H_2} - P_{\ker A^*} \in \mathscr{F}(H_2) \tag{8.4}$$

4. We also have:

$$\mathrm{ind}\, A = \dim\ker A^* A - \dim\ker A A^* = -\,\mathrm{ind}\, A^* \tag{8.5}$$

9**. Definition: "Finite Rank Operator"**

We define the *finite rank operators* by

$$\mathscr{L}_{fr}(H_1, H_2) := \left\{ A \in \mathscr{L}(H_1, H_2) \;\middle|\; \dim\mathrm{im}\, A < \infty \right\} \tag{9.1}$$

10**. Remark:**

- The space of compact operators $\mathcal{K}(H_1, H_2)$, contains the norm closure of the space $\mathscr{L}_{fr}(H_1, H_2)$. As we are only concerned with Hilbert spaces, it is even equal to that norm closure. This claim does not hold for general Banach spaces. Compare [Wer05, II.3.5].
- $\mathscr{L}_{fr}(H)$ and $\mathcal{K}(H)$ are two-sided *-ideals in $\mathscr{L}(H)$.

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11**. Lemma:**

Let $\dim H = \infty$ and $J \subset \mathscr{L}(H)$ be a two-sided ideal. If J contains a non-compact operator, then $J = \mathscr{L}(H)$.

Proof (11) **Hint:** You need the polar decomposition, $A = U|A|, |A| = \sqrt{A^* A}$ for $A \in C(H)$ and also the spectral theorem (see e.g. [Dav95, Chapter 2] and [RS80, Theorem VII.8]):

$$\begin{gathered} A = A^* \in C(H) \Leftrightarrow \exists E_A(\lambda))_{\lambda\in\mathbb{R}} \text{ orth.proj., increasing :} \\ \operatorname*{s-lim}_{x\to-\infty} E_A(\lambda) = 0, \quad \operatorname*{s-lim}_{\lambda\to+\infty} E_A(\lambda) = \mathrm{id}, \quad E_A(\lambda_0) = \operatorname*{s-lim}_{\lambda\to\lambda+} E_A(\lambda) \end{gathered} \tag{11.1}$$

s.t.

$$x \in \mathrm{dom}\, A \Leftrightarrow \int_{-\infty}^{+\infty} \lambda^2 \mathrm{d}\|E_A(\lambda)x\|^2 < \infty \tag{11.2}$$

$$\langle Ax, y\rangle = \int_{-\infty}^{+\infty} \lambda \mathrm{d}\langle E_A(\lambda)x, y\rangle \tag{11.3}$$

In particular $E_A(\lambda)A = AE_A(\lambda)$.

12. **Theorem: Spectrum of Compact Operators**

Let $A \in \mathcal{K}(H)$ with $\dim H = \infty$, then

1. $\operatorname{spec} A = (\lambda_j)_{j=1}^{\infty} \cup \{0\}$ where $\lambda_j \neq 0$, $|\lambda_j| \leq |\lambda_{j+1}|$ (counted with multiplicity).
2. For each $\lambda \in \operatorname{spec} A \setminus \{0\}$ there is a projection $P_\lambda \in \mathscr{L}_{fr}(H) \Leftrightarrow \dim P_\lambda < \infty$ s.t.

$$\operatorname{spec} P_\lambda A = \operatorname{spec} P_\lambda A = \{\lambda\} \tag{12.1}$$

$$\Rightarrow P_\lambda A = \lambda P_\lambda + D_\lambda \text{ with } D_\lambda \text{ nilpotent} \tag{12.2}$$

For $\lambda \neq \lambda' \in \operatorname{spec} A$, $P_\lambda P_{\lambda'} = 0$ and

$$P_\lambda = -\frac{1}{2\pi i} \oint_{|\zeta|=\varepsilon} (A-\zeta)^{-1} \mathrm{d}\zeta \tag{12.3}$$

3. $\|A\| = |\lambda_1|$

13. **Remark: Trivial Spectrum**

It may happen that $\operatorname{spec} A = \{0\}$, even though A is not of finite rank. Take for example in $L^2[0,1]$ the Volterra operator

$$Ax(t) := \int_0^t x(s)\mathrm{d}s \tag{13.1}$$

14. **Theorem: Property of Compact Operators**

1. $\mathcal{K}(H)$ is a two-sided norm-closed ideal.
2. $A \in \mathcal{K}(H) \Rightarrow A^* \in \mathcal{K}(H)$ i.e. $\mathcal{K}(H)$ is a *-ideal.
3. $\mathscr{L}(H)/\mathcal{K}(H)$ is a unital Banach-*-algebra called the *Calkin-algebra*. Let π be the corresponding injection from $\mathscr{L}(H)$. Then $T \in \mathscr{L}(H)$ is in $\mathscr{F}(H)$ iff $\pi(T)$ is invertible in the Calkin-algebra. The index characterizes the connected components of invertible elements in $\mathscr{L}/\mathcal{K}$.

$$\|\pi(T)\|_{\mathscr{L}/\mathcal{K}} = \inf_{K \in \mathcal{K}(H)} \|T+K\|_{\mathscr{L}(H)} \tag{14.1}$$

15. **Theorem:** FREDHOLM

If $K \in \mathcal{K}(H)$ then $\mathrm{id} + K \in \mathscr{F}(H)$ with $\operatorname{ind}(\mathrm{id} + K) = 0$

16. **Example: Possible Values for the Fredholm Index**

Look at the Hilbert space

$$\ell^2(\mathbb{C}) = \left\{ x \colon \mathbb{Z}_+ \to \mathbb{C} \;\middle|\; \|x\|_2^2 := \sum_{j \geq 0} |x_j|^2 < \infty \right\} \tag{16.1}$$

and consider the operator

$$(Sx)_j := \begin{cases} 0 & j = 0 \\ x_{j-1} & j > 0 \end{cases} \tag{16.2}$$

It is called the *unilateral shift* and is Fredholm of index -1. Then S^n has index $-n$ and $S^{*,n}$ has index n. This means there are Fredholm operators of any integer index.

17. **Theorem:** ATKINSON
$A \in \mathcal{C}(H_1, H_2)$ is Fredholm iff there are $R \in \mathscr{L}(H_2, H_1)$ and $K_i \in \mathcal{K}(H_i)$ such that

$$\mathrm{id}_{H_1} - RA = K_1 \quad \wedge \quad \mathrm{id}_{H_2} - AR = K_2 \tag{17.1}$$

Such an operator R is called a *parametrix* for A.

I.2. Traces and Determinants

For further information see for example [Kat95].

I.2.1. Finite Dimensional Spaces

To see what can be done, we first of all look at the situation of finite dimensional Hilbert spaces H. Afterwards this approach should provide some inspiration when we concern ourselves with the general case. For now we write $N := \dim H$.

18. **Definition: "Trace"**
So let $A \in \mathscr{L}(H)$, (e_i) a basis and (e^i) the corresponding dual basis. We will write $A_i^j := e^i(Ae_j)$. Then the *trace* of A is defined as

$$\operatorname{tr} A := \sum_{j=1}^{N} e^j(Ae_j) = \sum_j A_j^j \tag{18.1}$$

19. **Remark: Properties of the Trace**
   1. The trace $\operatorname{tr}\colon \mathscr{L}(H) \to \mathbb{C}$ is linear.
   2. It is invariant under permutations in $\mathscr{L}(H)$: $\operatorname{tr} AB = \operatorname{tr} BA$. That means, if $[A, B]$ denotes the usual commutator, $\ker \operatorname{tr} \supset [\mathscr{L}(H), \mathscr{L}(H)]$.

20. **Theorem: Vanishing Trace**
For any linear operator $A \in \mathscr{L}(H)$:

$$\operatorname{tr} A = 0 \Rightarrow \exists B, C \in \mathscr{L}(H) \colon A = [B, C]. \tag{20.1}$$

So in addition to the above property even

$$\ker \operatorname{tr} = [\mathscr{L}(H), \mathscr{L}(H)]. \tag{20.2}$$

21. **Discussion: A Better Description of the Trace**

Recall that the *spectrum* is defined as

$$\operatorname{spec} A := \{ \lambda \in \mathbb{C} \mid A - \lambda \text{ not invertible} \}, \tag{21.1}$$

but can also be expressed as $\operatorname{spec} A = \chi_A^{-1}(0)$ where $\chi_A\colon \mathbb{C} \ni \lambda \mapsto \det(A-\lambda) \in \mathbb{C}$ is the *characteristic polynomial* of A. Now it is possible to decompose A as

$$A = \sum_{\lambda \in \operatorname{spec} A} \lambda P_\lambda + D_\lambda, \tag{21.2}$$

where

$$P_\lambda = P_\lambda^2, \quad P_\lambda P_{\lambda'} = \delta_{\lambda\lambda'} P_{\lambda'}, \quad \sum P_\lambda = \mathrm{id}_H \tag{21.3}$$

$$P_\lambda D_\lambda = D_\lambda P_\lambda = D_\lambda, \quad \forall \lambda \exists n\colon D_\lambda^n = 0 \tag{21.4}$$

in particular

$$P_\lambda = \frac{1}{2\pi i} \oint_{|\zeta-\lambda|=\varepsilon} (A-\zeta)^{-1} \mathrm{d}\zeta \tag{21.5}$$

$$m_{geom}(\lambda) := \dim\ker(A-\lambda) \tag{21.6}$$

$$m_{alg}(\lambda) := \dim \operatorname{im} P_\lambda = \text{ multiplicity of } \lambda \text{ as a zero of } \chi_A \tag{21.7}$$

22. **Definition: "Chern Polynomial"**

Similar to χ_A we define the polynomial expansion

$$\det(\mathrm{id} + zA) =: \sum_{j=0}^{N} c_j(A) z^j \tag{22.1}$$

and call $c_j(A)$ the j-th *Chern polynomial*. Now the sum

$$c(A) := c_0(A) + c_1(A) + \ldots + c_n(A) \tag{22.2}$$

is the *total Chern polynomial*.

23. **Remark:**

The c_j are homogeneous polynomials of degree j in the eigenvalues of A and the most prominent ones are

$$c_0(A) = 1, \tag{23.1}$$

$$c_1(A) = \operatorname{tr} A \quad \text{and} \tag{23.2}$$

$$c_N(A) = \det A. \tag{23.3}$$

They are related to the *characteristic polynomial* via

$$\det(\mathrm{id} + zA) = z^N \det(A - (-\frac{1}{z})\,\mathrm{id}) = z^N \chi_A(-\frac{1}{z}) = \prod_{\lambda \in \operatorname{spec} A} (\lambda_j z + 1) \tag{23.4}$$

24. Lemma: Invariance

The Chern polynomials are invariant under the action of $\mathscr{L}^\times(H) \simeq \mathrm{GL}(N,\mathbb{C})$ on $\mathscr{L}(H)$ by conjugation. Let $T \in \mathscr{L}^\times(H)$ and $A \in \mathscr{L}(H)$, then

$$c_j(T^{-1}AT) = c_j(A). \tag{24.1}$$

This is a generalization of Remark 19 item 2 (Properties of the Trace).

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25. Problem:

Write a formula for $c_j(A)$ for $A = (A^i_j) \in \mathrm{Mat}(N,\mathbb{C})$.
Write an operator theoretic formula!

26. Lemma:

Any linear operator $A \in \mathscr{L}(H)$ induces a homomorphism of graded algebras $\Lambda^j A \in \mathscr{L}(\Lambda^j H)$ such that

$$\Lambda^j A(v_1 \wedge \ldots \wedge v_j) = (Av_1) \wedge \ldots \wedge (Av_j). \tag{26.1}$$

Then

$$c_j(A) = \mathrm{tr}_{\Lambda^j H} \Lambda^j A. \tag{26.2}$$

Proof (26) First of all we will denote index sets by

$$I^j_N := \left\{ I \subset \mathbb{N}_N \,\middle|\, \#I = j \right\}, \tag{26.3}$$

so that in particular $\#I^j_N = \binom{j}{N}$. Then we define the *diagonal minors* $A_I \in \mathscr{L}(H)$ by

$$A_I e_j := \begin{cases} Ae_j = j\text{-th column of } A & \text{if } j \in I \text{ and} \\ e_j & \text{otherwise.} \end{cases} \tag{26.4}$$

This yields

$$\det(\mathrm{id} + zA) = \sum_{j=0}^{N} z^j \sum_{I \in I^j_N} \det A_I, \tag{26.5}$$

so that

$$c_j(A) = \sum_{I \in I^j_N} \det A_I. \tag{26.6}$$

Next we have to compute $\mathrm{tr}\, \Lambda^j A$. An orthonormal basis of $\Lambda^j H$ is given by $(e_{i_1} \wedge \ldots \wedge e_{i_j})_{I \in I^j_N}$. So with

$$\mathrm{tr}\, \Lambda^j A = \sum_{I \in I^j_N} \langle \Lambda^j A(e_{i_1} \wedge \ldots \wedge e_{i_j}), e_{i_1} \wedge \ldots \wedge e_{i_j} \rangle \tag{26.7}$$

and

$$\langle Ae_{i_1} \wedge \dots \wedge Ae_{i_j}, e_{i_1} \wedge \dots \wedge e_{i_j} \rangle \tag{26.8}$$

$$= \sum_{\sigma \in S_j} A_{i_1}^{i_{\sigma(1)}} A_{i_2}^{i_{\sigma(2)}} \dots A_{i_j}^{i_{\sigma(j)}} \langle e_{i_{\sigma(1)}} \wedge \dots \wedge e_{i_{\sigma(j)}}, e_{i_1} \wedge \dots \wedge e_{i_{\sigma(j)}} \rangle \tag{26.9}$$

$$= \sum_{\sigma \in S_j} A_{i_1}^{i_{\sigma(1)}} A_{i_2}^{i_{\sigma(2)}} \dots A_{i_j}^{i_{\sigma(j)}} \operatorname{sgn} \sigma = \det A_I \tag{26.10}$$

the proof is complete.

27**. Theorem: Determinant Expansion**

The determinant can be written as a polynomial:

$$\det(\mathrm{id}_H + zA) = \sum_{j=0}^{N} z^j \operatorname{tr} \Lambda^j A = \prod_\lambda (1 + z\lambda(A)) \tag{27.1}$$

and the tr $\Lambda^j A$ are the *elementary symmetric polynomials* in spec A denoted as $\sigma_0, \sigma_1, \dots, \sigma_n$.

28**. Theorem:**

Any $p \in \mathbb{C}[z_1, \dots, z_N]$ which is symmetric, i.e. invariant under the action of S_N, can be written as a polynomial $\tilde{p} \in \mathbb{C}[\sigma_1, \dots, \sigma_N]$.

Hence the following polynomials can be written as polynomials in the σ_j.

29**. Definition: "Some Concrete Polynomials"**

1. *Adams polynomials*

$$a_j(A) := \operatorname{tr} A^j, \qquad j = 0, \dots, N \tag{29.1}$$

2. the *Chern character*

$$\operatorname{ch} A := \operatorname{tr} e^A \tag{29.2}$$

(not a polynomial)

3. the *Todd polynomial*

$$\operatorname{Td}(A) := \det\left(\frac{A}{\mathrm{id} - e^{-A}}\right) = \det\left(\sum_{j \geq 0} \frac{B_j}{j!} (-1)^j A^j\right) \tag{29.3}$$

4. the $\hat{A}$*-polynomial*

$$\hat{A}(A) = \det{}^{\frac{1}{2}}\left(\frac{A}{\sinh A}\right) = e^{\operatorname{tr} \log \frac{A}{\sinh A}} \tag{29.4}$$

5. the *L-polynomial*

$$L(A) = \det{}^{\frac{1}{2}}\left(\frac{A}{\tanh A}\right) \tag{29.5}$$

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30. **Theorem:**

Denote by $\mathcal{P}_{inv}(\mathscr{L}(\mathbb{C}^N))$ the S_N-invariant polynomials with values in $\mathbb{C}$; they form a ring. Then

$$\mathcal{P}_{inv}(\mathscr{L}(\mathbb{C}^N)) = \mathbb{C}[1, c_1, \dots, c_N] \tag{30.1}$$

i.e. any of these polynomials can be written as a polynomial in the Chern polynomials.

I.2.2. Infinite Dimensional Spaces

Extension to infinite dimension The only similarity in infinite dimensions is, that compact operators K have a spectrum $\operatorname{spec} K = \{\lambda_n \in \mathbb{C} \setminus \{0\}\} \cup \{0\}$ where $|\lambda_n| \geq |\lambda_{n+1}|$ with multiplicities ($m_{geo}(\lambda)$ and $m_{alg}(\lambda)$) and $\lim_{n\to\infty} |\lambda_n| = 0$. But now $\sum_n \lambda_n$ does not necessarily converge! In the (infinite dimensional) compact case we can also do the resolvent analysis as in Discussion 21 (A Better Description of the Trace), but now D_λ is only *quasi-nilpotent*, i.e.

$$\left(\|D_\lambda^n\|\right)^{\frac{1}{n}} \xrightarrow{n\to\infty} 0 \tag{I-1}$$

31. **Definition: "Hilbert-Schmidt Operator"**

$A \in \mathscr{L}(H_1, H_2)$ is called *Hilbert-Schmidt* (*HS*) iff

$$\sum_j \|Ae_j\|^2 < \infty \qquad \text{for any onb } (e_j) \text{ of } H_1. \tag{31.1}$$

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32. **Lemma:**

1. The HS-operators form a $\mathbb{C}$-vector space in $\mathcal{K}(H_1, H_2)$, called $\mathcal{K}_2(H_1, H_2)$.
2. $\|A\|_2^2 := \sum_j \|Ae_j\|^2$ does not depend on the choice of the orthonormal basis (e_j) and defines a norm on $\mathcal{K}_2(H_1, H_2)$.
3. For $K \in \mathcal{K}_2(H_1, H_2)$ and $A \in \mathscr{L}(H_2, H_1)$, also $K^* \in \mathcal{K}_2(H_2)$ and
$$AK \in \mathcal{K}_2(H_1) \quad \text{and} \quad \|AK\|_2 \leq \|A\| \, \|K\|_2 \tag{32.1}$$
similar for KA, i.e. $\mathcal{K}_2$ is a left and right module over respective bounded operators.
4. For $A, B \in \mathcal{K}_2(H)$ we can define a scalar product
$$\langle A, B\rangle_2 := \sum_{j=1}^{\infty} \langle Ae_j, Be_j\rangle \qquad \text{for any onb } (e_j) \tag{32.2}$$
$$\text{s.t. } |\langle A, B\rangle_2| \leq \|A\|_2 \|B\|_2 \tag{32.3}$$
5. If $A = A^* \in \mathcal{K}_2(H)$ then
$$\|A\|_2^2 = \sum_{\lambda \in \operatorname{spec} A} \lambda^2 \tag{32.4}$$

Proof (32) Let (e_j), (f_k) be orthonormal bases of H, $A \in \mathcal{K}_2(H)$. Then

$$\sum_{j\in\mathbb{N}} \|Ae_j\|^2 = \sum_{k,j\in\mathbb{N}} |\langle Ae_j, f_k\rangle|^2 = \sum_{j,k} |\langle e_j, A^* f_k\rangle|^2 = \sum_k \|A^* f_k\|^2 \tag{32.5}$$

33**. Theorem: Line Bundles**

Let (M, g^{TM}) be a closed oriented Riemannian manifold and $(E, h^E) \to M$ a Hermitian line bundle. Let moreover $k \in L^2(M \times M, \mathscr{L}(E \boxplus E))$ then the operator

$$Ks(p) := \int_M k(p,q)\{s(q)\}\,\mathrm{vol}_M(q) \tag{33.1}$$

is a HS operator in $\mathscr{L}(L^2(M, E))$ with

$$\|K\|_2^2 = \|k\|^2_{L^2(M\times M, \mathscr{L}(E\boxplus E))} = \int_{M\times M} \|k(p,q)\|^2_{\mathscr{L}(E_q, E_p)}\,\mathrm{vol}_{M\times M}(p,q) \tag{33.2}$$

Now back to the general case $H = H^+ \oplus H^-$ with involution $\alpha = \alpha^H$. Wen want to find operators with finite trace. If $(e_j)_{j\in\mathbb{N}}$ is an orthonormal basis, then we should have

34**. Definition: "Operators with Trace"**

A has a *trace* iff

$$\sum_{j\in\mathbb{N}} \langle Ae_j, e_j\rangle \text{ is finite} \tag{34.1}$$

and independent of the choice of $(e_j)_{j\in\mathbb{N}}$.

35**. Definition: "Trace Class"**

The set of linear operators

$$\mathcal{K}_1(H) := \left\{ A \in \mathscr{L}(H) \middle| A = \sum_{j=1}^{L} B_j C_j \text{ with } B_j, C_j \in \mathcal{K}_2(H) \right\} \tag{35.1}$$

is called the *trace class* of H.

36**. Theorem:**

1. Let $A \in \mathcal{K}_1(H)$ and (e_j) an orthonormal basis of H then

$$\sum_j |\langle Ae_j, e_j\rangle| < \infty. \tag{36.1}$$

2. If $A = A^* \in \mathcal{K}(H)$ is compact then $A \in \mathcal{K}_1(H)$ if and only if

$$\|A\|_1 := \sum_{\lambda\in\mathrm{spec}\,A} |\lambda| < \infty. \tag{36.2}$$

37. **Theorem:**

1. $\mathcal{K}_1(H)$ is a norm-closed two-sided *-ideal in $\mathscr{L}(H)$.
2. We define the *trace* on $\mathcal{K}_1(H)$ by

$$\operatorname{tr}\left(\sum_{j=1}^{L} B_j A_j\right) := \sum_{j=1}^{L}\sum_{k=1}^{\infty} \langle A_j e_k, B_j^* e_k\rangle \tag{37.1}$$

P This is well-defined.

The trace is a (continuous) linear functional with

$$\operatorname{tr} A^* = \overline{\operatorname{tr} A}, \quad \operatorname{tr} AB = \operatorname{tr} BA \qquad \text{for } A \in \mathcal{K}_1(H), B \in \mathscr{L}(H) \tag{37.2}$$

3. If $K = K^* \in \mathcal{K}_1(H)$ with eigenvalues $(\lambda_j(K))$, then

$$\operatorname{tr} K = \sum_{j=1}^{\infty} \lambda_j(K) \tag{37.3}$$

Remark For $K \neq K^*$ this is still true, but not at all obvious. (e.g. look again at the Volterra operator). It was proved by LIDSKII in 1959.

P **Problem** Work out the trace on $\mathcal{K}_1(L^2(M,E))$ in the situation of Theorem 33 (Line Bundles).

Next Point: Show how to extent determinants to ∞ dimensions, needs $\|\cdot\|_{\mathcal{K}_1(H)}$.
Write $A = U|A|$ and define

$$\|A\|_{\mathcal{K}_1(H)} := \operatorname{tr}|A| \tag{I-2}$$

The good aspect of the result Theorem 27 (Determinant Expansion) is that it generalizes as it stands to infinite dimensions:

38. **Theorem:**

If $A \in \mathcal{K}_1(H)$ then

$$\det(I_H + zA) = \sum_{k\geq 0} z^k \operatorname{tr} \Lambda^k A = \prod_{\lambda_j}(1 + z\lambda_j(A)) \tag{38.1}$$

Main Example M closed (oriented) manifold, $E^+, E^- \to M$ smooth $\mathbb{C}$-vector bundles, $D\colon C^\infty(M,E^+) \to C^\infty(M,E^-)$ linear elliptic differential operator. Then we put metrics g^{TM} on TM and $h^{E^\pm}$ on $E^\pm$ to get Hilbert spaces $L^2(M,E^\pm)$ with scalar product

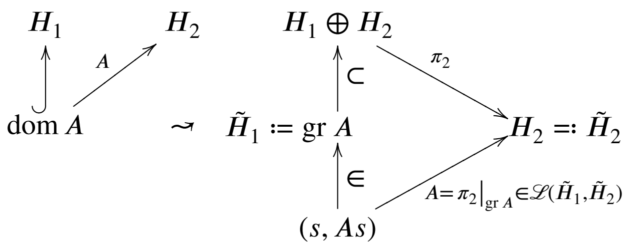
$$\left(s_1, s_2\right)_{L^2(M,E^\pm)} := \int_M h^{E^\pm}(s_1(p), s_2(p)) \operatorname{vol}_M(p) \tag{I-3}$$

$\bar{D}$ is a closed[1] unbounded Fredholm operator $H_1 := L^2(M,E^+) \to H_2 := L^2(M,E^-)$.

[1] closed always means densely defined

39. **Discussion: Reduction to Bounded Operators**

Instead of looking at an unbounded operators $A \in C(H_1, H_2)$, we can regard the graph of A as a Hilbert space, because A is closed. There π_2 corresponds to A.



(39.1)

Note that the identification of dom A and gr A induces the graph norm on dom A, which in general gives a finer topology than that induced from $\|\cdot\|_{H_1}$.

40. **Discussion: Polar Decomposition**

Let $A \in \mathscr{L}(H_1, H_2)$ and define

$$|A| := (A^* A)^{\frac{1}{2}} \in \mathscr{L}(H_1) \quad \textit{modulus} \text{ of } A \tag{40.1}$$

$|A|$ is self-adjoint and positive (i.e. $\forall x \in H_1 \colon \langle Ax, x\rangle \geq 0$).

41. **Theorem:**

There is an operator $U \in \mathscr{L}(H_1, H_2)$ s.t.

$$A = U|A| \tag{41.1}$$

Here UU^* is the orthogonal projection in H_2 onto $\overline{\text{im}\, A} = (\ker A^*)^{\perp}$ and U^*U is the orthogonal projection in H_1 onto $\overline{\text{im}\, A^*} = (\ker A)^{\perp}$.

Proof (41) Note first that $\ker A = \ker|A|$ since

$$\||A|x\|^2_{H_1} = \langle |A|x, |A|x\rangle_{H_1} = \langle |A|^2 x, x\rangle_{H_1} = \langle A^* Ax, x\rangle_{H_1} = \|Ax\|^2_{H_1} \tag{41.2}$$

Then the map $U \colon \text{im}|A| \ni |A|x \mapsto Ax \in \text{im}\, A$ is a well-defined isometry and satisfies

$$\forall x, y \in \text{im}|A| \colon \langle Ux, Uy\rangle_{H_2} = \langle x, y\rangle_{H_1} = \langle x, U^*Uy\rangle_{H_1} \tag{41.3}$$

$$\Rightarrow U^*U\big|_{\text{im}|A|} = I_{\text{im}|A|} \tag{41.4}$$

U extends to $\overline{\text{im}|A|} \to \overline{\text{im}\, A}$ by continuity. Put $U|_{\overline{\text{im}|A|}^{\perp}} = U|_{\ker|A|} := 0$. U is a partial isometry.

42. **Corollary: Generalized Inverse**

If $A \in \mathscr{F}(H_1, H_2)$, then $\exists \delta > 0 \colon \||A|x\| = \|Ax\| \geq \delta\|x\| \forall x \perp \ker A$.
Hence $|A|$ (and A) admits a *generalized inverse* $|A|^{-1} \in \mathscr{F}(H_1)$

$$|A|^{-1}x := \begin{cases} y & \text{if } x = |A|y,\ x \perp \ker|A| \\ 0 & \text{if } x \in \ker|A| \end{cases} \tag{42.1}$$

Let $A \in \mathscr{F}(H_1, H_2)$ and write $A = U|A|$ to get the parametrix $R = |A|^{-1}U^*$ then

$$AR = UU^* = P_{\overline{\operatorname{im} A}} \tag{I-4}$$
$$I_{H_2} - P_{\overline{\operatorname{im} A}} = P_{(\operatorname{im} A)^\perp} \in \mathcal{K}_1(H_2) \tag{I-5}$$
$$\operatorname{tr}_{H_2}(I_{H_2} - AR) = \dim \operatorname{coker} A \tag{I-6}$$
$$RA = U^*U = P_{(\ker A)^\perp} \tag{I-7}$$
$$I_{H_1} - P_{(\ker A)^\perp} = P_{\ker A} \in \mathcal{K}_1(H_1) \tag{I-8}$$
$$\operatorname{tr}_{H_1}(I_{H_1} - RA) = \dim \ker A \tag{I-9}$$

43. **Theorem: Good Parametrices**

$A \in \mathscr{L}(H_1, H_2)$ is Fredholm iff there is $R \in \mathscr{L}(H_2, H_1)$ s.t.

$$I_{H_1} - RA \in \mathcal{K}_1(H_1)$$
$$I_{H_2} - AR \in \mathcal{K}_1(H_2)$$
$$\operatorname{ind} A = \operatorname{tr}_{H_1}(I_{H_1} - RA) - \operatorname{tr}_{H_2}(I_{H_2} - AR) \tag{43.1}$$

If Equation 43.1 holds, then R is called a *good parametrix*. In fact any parametrix, which satisfies the first two relations is already good. To prove this we have to use the following:

P 44. **Lemma:**

Let $A \in \mathscr{L}(H_1, H_2)$, $B \in \mathscr{L}(H_2, H_1)$ such that AB, BA are of trace class, then

$$\operatorname{tr}_{H_1} BA = \operatorname{tr}_{H_2} AB \tag{44.1}$$

This follows for example with 3.

Consider a closed operator $D \in \mathscr{F}_{cl}(H_1, H_2)$, i.e. D may not extend to $\mathscr{L}(H_1, H_2)$

45. **Definition: "Discrete Operator"**

D is called *discrete* iff $j\colon \operatorname{dom} D \hookrightarrow H_1$ is compact.

P 46. **Lemma:**

If D is self-adjoint then spec D is discrete, i.e. consists only of isolated eigenvalues of finite multiplicity, iff D is discrete in the sense of the former definition.

47. **Theorem:** McKean-Singer

Assume that $D = D^*$ is discrete in H and for $t > 0$, e^{-tD^2} with eigenvalues $(e^{-t\lambda_j^2})_{\lambda_j \in \operatorname{spec} D}$. If in addition D anti-commutes with an involution α, i.e.

$$\alpha(\operatorname{dom} D) = \operatorname{dom} D \quad \wedge \quad \alpha D + D\alpha = 0 \tag{47.1}$$

Then $H = H^+ \oplus H^-$, $\operatorname{dom} D = (\operatorname{dom} D)^+ \oplus (\operatorname{dom} D)^- =: (\operatorname{dom} D^+) \oplus (\operatorname{dom} D^-)$ and $D = \left(\begin{smallmatrix} 0 & D^- \\ D^+ & 0 \end{smallmatrix}\right)$. Also D^+ is a Fredholm operator in $\mathscr{F}_{cl}(H^+, H^-)$ with adjoint D^- and

$$\operatorname{ind} D^+ = \operatorname{tr}_H(\alpha e^{-tD^2}) = \operatorname{tr}_{H^+} e^{-tD^-D^+} - \operatorname{tr}_{H^-} e^{-tD^+D^-} \tag{47.2}$$

Explanation

$$\phi_j = \phi_j^+ + \phi_j^- \qquad \alpha\phi_j = \phi_j^+ - \phi_j^- \tag{47.3}$$

$$\begin{aligned} \operatorname{tr}_H(\alpha e^{-tD^2}) &= \sum_j \langle \alpha e^{-tD^2}\phi_j, \phi_j\rangle = \sum_j e^{-t\lambda_j^2}\langle \alpha\phi_j, \phi_j\rangle \\ &= \sum_j e^{-t\lambda_j^2}(\|\phi_j^+\|^2 - \|\phi_j^-\|^2) \end{aligned} \tag{47.4}$$

$$\frac{d}{dt}\operatorname{tr}_H \alpha e^{-tD^2} = -\operatorname{tr}_H(\alpha D^2 e^{-tD^2}) = -\operatorname{tr}_H(D\alpha D e^{-tD^2}) = 0 \tag{47.5}$$

P

48. **Problem:**
D^+D^- and D^-D^+ have the same non-zero eigenvalues

I.3. Properties of the Fredholm Index

Let $A \in \mathscr{F}_{bd}(H_1, H_2)$ for any Hilbert spaces H_1, H_2. And let $R \in \mathscr{L}(H_2, H_1)$ be a good parametrix, i.e.

$$I_{H_1} - RA \in \mathcal{K}_1(H_1) \tag{I-10}$$

$$I_{H_2} - AR \in \mathcal{K}_1(H_2) \tag{I-11}$$

$$\operatorname{ind} A = \operatorname{tr}_{H_1}(I_{H_1} - RA) - \operatorname{tr}_{H_2}(I_{H_2} - AR) \tag{I-12}$$

49. **Theorem: Properties of the Index**

1. *stability*, i.e. not disturbed by small perturbations:
   If $B \in \mathscr{L}(H_1, H_2)$ s.t.
   $$\|BR\|_{H_2} < 1, \quad \|RB\|_{H_1} < 1 \tag{49.1}$$
   then
   $$A - B \in \mathscr{F}_{bd}(H_1, H_2) \quad \text{with} \quad \operatorname{ind}(A - B) = \operatorname{ind} A \tag{49.2}$$
2. *Logarithmic law*
   $$\begin{gathered} H_1 \underset{R_1}{\overset{A_1}{\rightleftarrows}} H_2 \underset{R_2}{\overset{A_2}{\rightleftarrows}} H_3 \qquad A_j \in \mathscr{F}_{bd}(H_{j-1}, H_j), j = 2, 3, \\ (R_j \in \mathscr{L}(H_j, H_{j-1}) \text{ good parametrices of } A_j\ ), j = 1, 2 \end{gathered} \tag{49.3}$$
   then
   $$A_2A_1 \in \mathscr{F}(H_1, H_3) \quad \text{with} \quad \operatorname{ind} A_1A_2 = \operatorname{ind} A_1 + \operatorname{ind} A_2 \tag{49.4}$$

Proof (49)

1. Test case: A is invertible, $R = A^{-1}$, then $A - B$ is as good as A if it is invertible:

$$(\underbrace{A - B}_{A_1})^{-1} = (A(I_{H_1} - A^{-1}B))^{-1} = (I_{H_1} - RB)^{-1}R \tag{49.5}$$

$$= \sum_{j\geq 0}(RB)^j R = R\sum_{j\geq 0}(BR)^j =: R_1 \tag{49.6}$$

Try now R_1 as a (good) parametrix for A_1 in general.

$$I_{H_1} - R_1A_1 = I_{H_1} - \sum_{j\geq 0}(RB)^j R(A - B) = I_{H_1} - \sum_{j\geq 0}(RB)^j RA - \sum_{j\geq 0}(RB)^{j+1} \tag{49.7}$$

$$= I_{H_1} - RA - \sum_{j\geq 0}(RB)^{j+1}RA + \sum_{j\geq 0}(RB)^{j+1} \tag{49.8}$$

$$= \left(\sum_{j\geq 0}(RB)^{j+1} + I\right)\left(I_{H_1} - RA\right) = \sum_{j=0}^{\infty}(RB)^j(I - RA) \tag{49.9}$$

P

Compute also

$$I_{H_2} - A_1R_1 = (I_{H_2} - AR)\sum_{j\geq 0}(BR)^j \tag{49.10}$$

$$\begin{aligned}&\operatorname{tr}_{H_1}(I_{H_1} - R_1A_1) - \operatorname{tr}_{H_2}(I_{H_2} - A_1R_1)\\ &\quad = \operatorname{ind} A + \underbrace{\operatorname{tr}_{H_1}\sum_{j\geq 1}(RB)^j(I_{H_1} - RA) - \operatorname{tr}_{H_2}\sum_{j\geq 1}(I_{H_2} - AR)(BR)^j}_{=0}\end{aligned} \tag{49.11}$$

2. Set $A := A_2A_1$ then A is Fredholm (think of Lemma 6 (Semi-Fredholm Condition)) and we expect $R_1R_2 =: R$ to be a good parametrix.

Compute

$$\begin{aligned}I_{H_1} - RA = I_{H_1} - R_1R_2A_2A_1 &= I_{H_1} + R_1(I_{H_2} - R_2A_2)A_1 - R_1A_1\\ &= (I_{H_1} - R_1A_1) + R_1(I_{H_2} - R_2A_2)A_1\end{aligned} \tag{49.12}$$

$$\begin{aligned}\Rightarrow \operatorname{tr}_{H_1}(I_{H_1} - RA) &= \operatorname{tr}_{H_1}(I_{H_1} - R_1A_1) + \operatorname{tr}_{H_2}(I_{H_2} - R_2A_2)A_1R_1\\ &= \operatorname{tr}_{H_1}(I_{H_1} - R_1A_1) + \operatorname{tr}_{H_2}(I_{H_2} - R_2A_2)\\ &\quad - \operatorname{tr}_{H_2}(I_{H_2} - R_2A_2)(I_{H_2} - A_1R_1)\end{aligned} \tag{49.13}$$

$$\Rightarrow \operatorname{ind} A_1A_2 = \operatorname{ind} A_1 + \operatorname{ind} A_2 \tag{49.14}$$

50. **Definition: "Operator Product"**

Let $A^j \in \mathscr{F}(H_1^j, H_2^j)$ for $j = 1, 2$. Then we can construct the *Hilbert space tensor product* $H_1^1 \otimes H_1^2$ by taking the completion of the algebraic tensor product

$$\left\{ \sum_{k=1}^{N} x_k^1 \otimes x_k^2 \;\middle|\; x_k^1 \in H_1^1, x_k^2 \in H_1^2, N \in \mathbb{N} \right\} \tag{50.1}$$

under the pre-Hilbert structure defined by

$$\langle \sum_k x_k^1 \otimes x_k^2, \sum_l x_l^1 \otimes x_l^2 \rangle := \sum_{k,l} \langle x_k^1, x_l^1 \rangle \langle x_k^2, x_l^2 \rangle \tag{50.2}$$

Then we define the following maps:

$$\begin{array}{ccc} H_1^1 \otimes H_1^2 & \xrightarrow{A^1 \otimes I_{H_1^2}} & H_2^1 \otimes H_1^2 \\ \oplus & & \oplus \\ H_2^1 \otimes H_2^2 & \xrightarrow[A^{1,*} \otimes I_{H_2^2}]{} & H_1^1 \otimes H_2^2 \end{array} \tag{50.3}$$

$$\mathcal{A} := \begin{pmatrix} A^1 \otimes I_{H_1^2} & -I_{H_2^1} \otimes A^{2,*} \\ I_{H_1^1} \otimes A^2 & A^{1,*} \otimes I_{H_2^2} \end{pmatrix} \in \mathscr{L}(H_1^1 \otimes H_1^2 \oplus H_2^1 \otimes H_2^2, H_2^1 \otimes H_1^2 \oplus H_1^1 \otimes H_2^2) \tag{50.4}$$

$$\Rightarrow \mathcal{A}^* = \begin{pmatrix} A^{1,*} \otimes I_{H_1^2} & I_{H_1^1} \otimes A^{2,*} \\ -I_{H_2^1} \otimes A^2 & A^1 \otimes I_{H_2^2} \end{pmatrix} : \begin{pmatrix} H_2^1 \otimes H_1^2 \\ H_1^1 \otimes H_2^2 \end{pmatrix} \to \begin{pmatrix} H_1^1 \otimes H_1^2 \\ H_2^1 \otimes H_2^2 \end{pmatrix} \tag{50.5}$$

Next we compute:

$$\mathcal{A}^*\mathcal{A} = \begin{pmatrix} A^{1,*}A^1 \otimes I_{H_1^2} \oplus I_{H_1^1} \otimes A^{2,*}A^2 & 0 \\ 0 & I_{H_2^1} \otimes A^2 A^{2,*} \oplus A^1 A^{1,*} \otimes I_{H_2^2} \end{pmatrix} \tag{50.6}$$

We want to show that this operator is again Fredholm. So what is the kernel of $A^{1,*}A^1 \otimes I_{H_1^2} \oplus$
P $I_{H_1^1} \otimes A^{2,*}A^2$?

In the end we get $\mathcal{A}$ is Fredholm with $\operatorname{ind} \mathcal{A} = \operatorname{ind} A^1 \operatorname{ind} A^2$.

I.4. Pseudodifferential Operators (ψdo)

I.4.1. Euclidean Case

For simplicity look at the model space $\Psi\mathrm{DO}(\mathbb{R}^m, \mathbb{C}^k)$.

Recall the *Schwartz space*

$$\mathcal{S}(\mathbb{R}^m, \mathbb{C}^k) = \left\{ s \in C^\infty(\mathbb{R}^m, \mathbb{C}^k) \;\middle|\; |(1+|x|^2)^{\frac{k}{2}} D_x^\alpha s(x)| \le C_{\alpha,\beta} \forall x \in \mathbb{R}^m \right\} \tag{I-13}$$

Then we have the *Fourier transform*

$$F\colon \mathcal{S} \ni s \mapsto (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} e^{-i\langle x,\xi\rangle} s(x) \mathrm{d}x \; =: \hat{s}(\xi) \in \mathcal{S} \tag{I-14}$$

P

and

$$\widehat{D_x^\alpha s}(\xi) = \xi^\alpha \hat{s}(\xi) \tag{I-15}$$

$$\widehat{x^\alpha s}(\xi) = D_\xi^\alpha \hat{s}(\xi) \tag{I-16}$$

$$(s_1, s_2)_{L^2} = (\hat{s}_1, \hat{s}_2)_{L^2} \tag{I-17}$$

$$F^{-1}s(x) = (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} e^{+i\langle x,\xi\rangle} s(\xi) \mathrm{d}\xi \; = (Fs)(-x) \tag{I-18}$$

$$\begin{aligned} P(x,D)s(x) &= \sum_{|\alpha|\le l} A_\alpha(x) D_x^\alpha s(x) \\ &= (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} e^{i\langle x,\xi\rangle} \underbrace{\sum_{|\alpha|\le l} A_\alpha(x) \xi^\alpha \hat{s}(\xi)}_{\hat{P}(x,\xi)} \mathrm{d}\xi \; \in \mathrm{Diff}_l(\mathbb{R}^m, \mathbb{C}^k) \end{aligned} \tag{I-19}$$

Standard Assumption (controls growth at infinity)

$$|D_x^\beta D_\xi^\alpha P(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|^2)^{\frac{l}{2}-|\alpha|} \tag{I-20}$$

Now we get a *calculus*:
Diff$(\mathbb{R}^m, \mathbb{C}^k)$ is an algebra under concatenation and also a *-algebra, i.e.

$$(P(x,D)s_1, s_2)_{L^2} = (s_1, P(x,D)^\dagger s_2)_{L^2} \quad \text{with } P(x,D)^\dagger \in \mathrm{Diff}(\mathbb{R}^m, \mathbb{C}^k) \tag{I-21}$$

Here $P^\dagger \in \mathrm{Diff}_l \Leftrightarrow P \in \mathrm{Diff}_l$ and we take the L^2 scalar product induced by

$$\|s\|_{L^2(\mathbb{R}^m,\mathbb{C}^k)} = \int_{\mathbb{R}^m} |s(x)|^2_{\mathbb{C}^k} \mathrm{d}x\,, \qquad s \in \mathcal{S}(\mathbb{R}^m, \mathbb{C}^k) \tag{I-22}$$

Principal Symbol

$$\hat{P}_{princ}(x_0, \xi_0)(z) := \frac{i^l}{l!} P(\phi^l s)(x_0), \tag{I-23}$$

where $\phi \in C_c^\infty(\mathbb{R}^m), \phi(x_0) = 0, \mathrm{d}\phi(x_0) = \xi_0$ and $s \in C_c^\infty(\mathbb{R}^m, \mathbb{C}^k)$ with $s(x_0) = z$ (I-24)

$$\Rightarrow \hat{P}_{princ}(x_0, \xi_0) = \sum_{|\alpha|=l} A_\alpha(x) \xi^\alpha \tag{I-25}$$

Now without requiring $\phi(x_0) = 0$, choose $\phi(x) = \langle x, \xi_0 \rangle$ and set $c := (2\pi)^{-\frac{m}{2}}$:

$$\begin{aligned}
\hat{P}_{princ}(x_0, \xi_0)z &= \lim_{t\to\infty} t^{-l} e^{-it\phi(x_0)} P(x, D)(e^{it\phi} s)(x_0) \\
&= \lim_{t\to\infty} c \int_{\mathbb{R}^m} e^{i\langle x_0,\xi\rangle - it\langle x_0,\xi_0\rangle} t^{-l} \hat{P}(x_0, \xi)(\widehat{e^{it\phi} s})(\xi) \mathrm{d}\xi \\
&= \lim_{t\to\infty} c \int e^{i\langle x_0,\xi - it\xi_0\rangle} t^{-l} \hat{P}(x, \xi)\hat{s}(\xi - it\xi_0)\mathrm{d}\xi \\
&= \lim_{t\to\infty} e^{-i\langle x_0, t\xi_0\rangle} c \int e^{i\langle x_0,\eta\rangle} t^{-l} \hat{P}(x_0, \eta + t\xi_0)\hat{s}(\pi)\mathrm{d}\eta \\
&= c \int e^{i\langle x,\eta\rangle} \hat{P}_{princ}(x, \xi_0)\hat{s}(\eta)\mathrm{d}\eta \ = \hat{P}_{princ}(x_0, \xi_0)s(x_0)
\end{aligned} \tag{I-26}$$

We did not use that $\hat{P}(x, \xi)$ is a polynomial, but only that

$$\lim_{t\to\infty} t^{-l} \hat{P}(x_0, \eta + t\xi_0) \xrightarrow{\text{in } L^2} P(x_0, \xi_0) \tag{I-27}$$

51. **Definition: "Symbol Space"**

We define the *symbol space* of order $l \in \mathbb{R}$, $\mathrm{Sym}_l(\mathbb{R}^m, \mathbb{C}^k)$, as the space of all $\hat{p} \in C^\infty(\mathbb{R}^m \times \mathbb{R}^m, \mathscr{L}(\mathbb{C}^k))$ such that Equation I-20 (Standard Assumption) holds.

52. **Lemma:**

P

Let $\hat{p} \in \mathrm{Sym}_l(\mathbb{R}^m, \mathbb{C}^k)$ and define

$$P(x, D)s(x) := (2\pi)^{\frac{-m}{2}} \int_{\mathbb{R}^m} e^{i\langle x,\xi\rangle} \hat{p}(x, \xi)\hat{s}(\xi)\mathrm{d}\xi \tag{52.1}$$

Then P defines an element of $\mathscr{L}(\mathcal{S}, \mathcal{S})$.

53. **Example: Crucial**

We write

$$\langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}} \in C^\infty(\mathbb{R}^m) \tag{53.1}$$

$$\langle \xi \rangle^s = (1 + |\xi|^2)^{\frac{s}{2}} \in C^\infty(\mathbb{R}^m) \tag{53.2}$$

P

Then even $\langle \xi \rangle^s \in \mathrm{Sym}_s(\mathbb{R}^m, \mathbb{C}^k)$ for all $s \in \mathbb{R}$.

$$\varLambda_s u(x) := (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} e^{i\langle x,\xi\rangle} \langle \xi \rangle^s \hat{u}(\xi)\mathrm{d}\xi \,, u \in \mathcal{S} \subset L^2 \tag{53.3}$$

P

Here $\varLambda_s$ is closable in $L^2(\mathbb{R}^m, \mathbb{C}^k)$ and essentially self-adjoint.

54. **Definition: "Classic Pseudodifferential Operator"**

We define the space of *classic pseudodifferential operators* of order l as

$$\Psi\mathrm{DO}_l(\mathbb{R}^m, \mathbb{C}^k) := \left\{ P(x, D) \;\middle|\; \hat{P} \in \mathrm{Sym}_l(\mathbb{R}^m, \mathbb{C}^k) \right\} \tag{54.1}$$

Now we define naturally $\Psi\mathrm{DO} := \cup_{l\in\mathbb{R}}\Psi\mathrm{DO}_l$ but while $\cap_{l\in\mathbb{Z}_+}\mathrm{Diff}_l = C^\infty(\mathbb{R}^m, \mathscr{L}(\mathbb{C}^k))$ now $\Psi\mathrm{DO}_{-\infty} := \cap_{l\in\mathbb{R}}\Psi\mathrm{DO}_l$ is a big space of non-trivial operators!

So what is the principal symbol of Λ_r? Following the previous recipe we have to compute the L^2-limit of

$$\lim_{t\to\infty} t^{-r\langle\pi+t\xi_0\rangle^r} = \lim_{t\to\infty} t^{-r}(1 + |\eta + t\xi_0|^2)^{\frac{r}{2}} \tag{I-28}$$

$$= \lim_{t\to\infty} t^{-r}(1 + |\eta|^2 + t^2|\xi_0|^2 - 2t\langle\eta, \xi_0\rangle)^{\frac{r}{2}} = \lim_{t\to\infty} (|\xi_0|^2 + \frac{1}{t^2} - \frac{2}{t}\langle\eta, \xi_0\rangle)^{\frac{r}{2}} \tag{I-29}$$

We may assume $|\eta| \leq R$ than the elements are bounded between $\frac{1}{2}|\xi_0|$ and $2|\xi_0|$ if $t \geq t(R)$ and thus the limit is $|\xi_0|^r$.

55. **Definition: "Formal Expansion"**

For $p \in \mathrm{Sym}_l(\mathbb{R}^m, \mathbb{C}^k)$ a *formal expansion* (or *formal development*) is a formal series $p_j \in \mathrm{Sym}_{n_j}(\mathbb{R}^m, \mathbb{C}^k)$ such that $n_j \to -\infty$ and if $p - \sum_{j=1}^r p_j \in \mathrm{Sym}_{s_j}(\mathbb{R}^m, \mathbb{C}^k)$ then $s_j \to -\infty$ too. We write:

$$p(x, \xi) \sim \sum_{j=1}^{\infty} p_j(x, \xi) \tag{55.1}$$

Additional Requirement If $p \in \mathrm{Sym}_l(\mathbb{R}^m, \mathbb{C}^k)$, $|\xi| >> 0$, there shall exist a *formal expansion*

$$p(x, \xi) \sim \sum_{j\in\mathbb{Z}_+} p_j(x, \xi) \tag{I-30}$$

such that $p - \sum_{j=1}^r p_j \in \mathrm{Sym}_{l-j}(\mathbb{R}^m, \mathbb{C}^k)$ and p_j is positive homogeneous in ξ of degree $l - j + 1$. Then of course $p_{pr} = p_1$.

Example For $|\xi| \geq 2$

$$(1 + |\xi|^2)^{-\frac{l}{2}} = |\xi|^{-l}\left(1 + \frac{1}{|\xi|^2}\right)^{-\frac{l}{2}} = |\xi|^{-l}\sum_{k\geq 0}\binom{-\frac{l}{2}}{k}\left(-\frac{1}{|\xi|^2}\right)^k \tag{I-31}$$

Fact: Formal expansions induce operator expansions modulo $\Psi\mathrm{DO}_{-\infty}(\mathbb{R}^m, \mathbb{C}^k)$.

56. **Theorem:**

1. $\Psi\mathrm{DO}(\mathbb{R}^m, \mathbb{C}^k)$ is an algebra, s.t.

$$\widehat{(P_1 P_2)}_{pr} = \hat{P}_{1,pr} \hat{P}_{2,pr} \tag{56.1}$$

2. $P \in \Psi\mathrm{DO} \Rightarrow P^\dagger \in \Psi\mathrm{DO}$

3. The map $P \mapsto \hat{P}_{pr}$ is surjective and $\hat{P}_{1,pr} = \hat{P}_{2,pr} \Rightarrow P_1 - P_2 \in \Psi\mathrm{DO}_{l-1}$.

I.4.2. Regularity Theory

57. **Definition: "**SOBOLEV **Space"**

We write

$$\begin{aligned} \operatorname{dom} \bar{\Lambda}_s &=: H^s(\mathbb{R}^m, \mathbb{C}^k) =: \textit{Sobolev space of order } s \\ &= \left\{ u: \mathbb{R}^m \to \mathbb{C}^k \;\middle|\; \langle\xi\rangle^s |\hat{u}(\xi)| \in L^2(\mathbb{R}^m, \mathbb{C}^k) \right\} \end{aligned} \tag{57.1}$$

58. **Lemma:**

The scalar product (s_1, s_2) for $s_1, s_2 \in \mathcal{S}$ extends by continuity to a non-degenerate bilinear form

$$(s_1, s_2)_s : H^s \times H^{-s} \to \mathbb{C} \tag{58.1}$$

i.e. it gives an isomorphism $H^{-s} \simeq H^{s,*}$.

H^t is also a Hilbert space itself with

$$\|s\|_t^2 = (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} \langle\xi\rangle^{it} |\hat{s}(\xi)|^2 \mathrm{d}\xi \tag{58.2}$$

59. **Theorem:**

If $P \in \Psi\mathrm{DO}_l$, then P extends by continuity to an element of $\mathscr{L}(H^r(\mathbb{R}^m, \mathbb{C}^k), H^{r-l}(\mathbb{R}^m, \mathbb{C}^k))$.

$\left(H^r(\mathbb{R}^m, \mathbb{C}^k)_{r \in \mathbb{R}}\right)$ is called the *Sobolev chain* of $\mathbb{R}^m \times \mathbb{C}^k$ (associated to Λ).

60. **Theorem:** SOBOLEV **Imbedding**

P $H^l(\mathbb{R}^m, \mathbb{C}^k) \subset C^n(\mathbb{R}^m, \mathbb{C}^k)$ if $l > n + \frac{m}{2}$.

61. **Theorem: L^2-boundedness**

P Let $P \in \Psi\mathrm{DO}_l$, then P extends to an element of $\mathscr{L}(L^2(\mathbb{R}^m, \mathbb{C}^k))$ iff $l \leq 0$.

62. **Theorem:** RELLICH

If $P \in \Psi\mathrm{DO}_l$ defined by $\hat{P}$ satisfies that $\pi_1 \operatorname{supp} P(x, \xi)$ is compact in $\mathbb{R}^m$ and $l < 0$, then P is compact.

63. **Lemma:**

$P \in \Psi\mathrm{DO}_{-\infty}$ is given by a smooth kernel K_p, i.e.

$$Ps(x) = \int_{\mathbb{R}^m} K_p(x,y)\{s(y)\}\mathrm{d}y \tag{63.1}$$

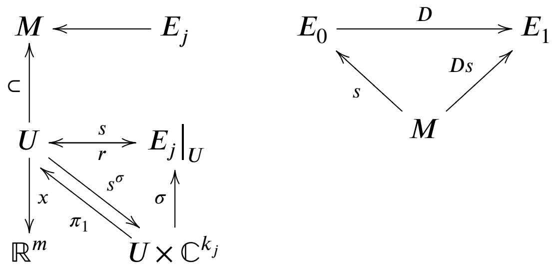
with all derivatives of K_p bounded.

For a differential operator we had *locality* i.e. $\operatorname{supp} Ps \subset \operatorname{supp} s$. But this is not true anymore for pseudodifferential operators.

I.4.3. Manifold Case

Fix M^m a closed oriented manifold of dimension m and on it a Riemannian metric g^{TM}. Choose also smooth $\mathbb{C}$-vector bundles $E_j \to M, j = 1, 2$ of rank k_j with Hermitian metrics h^{E_j}. So then we get the Hilbert spaces $H_j = L^2(M, E_j)$, i.e. for $s_j \in C(M, E_j)$ we have

$$\|s_j\|^2_{L^2(M,E_j)} = \int_M \|s_j(p)\|^2_{h_{E_j}} \operatorname{vol}_M(p) \tag{I-32}$$



(I-33)

Now we have *differential operators* $D\colon C^\infty(M, E_0) \to C^\infty(M, E_1)$ in local frames σ and coordinates x:

$$\sigma_1^{-1} Ds = (\sigma_1^{-1} D\sigma_0)s_{\sigma_0} = \sum_{|\alpha|\le l} A_\alpha D_x^\alpha s_{\sigma_0} \qquad \text{where } A \in C^\infty(U, \mathscr{L}(\mathbb{C}^{k_0}, \mathbb{C}^{k_1})) \tag{I-34}$$

$$D_{x^j} = -i\frac{\partial}{\partial x^j}, \quad D_x^\alpha = D_{x_1}^{\alpha_1} \dots D_{x_n}^{\alpha_n} \tag{I-35}$$

Recall the *principal symbol*: for $\xi \in \mathbb{R}^m$ we replace D_x^α by ξ^α for $|\alpha| = l$ to get

$$\sum_{|\alpha|=l} A_\alpha \xi^\alpha \quad \text{homogeneous polynomial in } \xi \tag{I-36}$$

(Seeley) Fix $\xi \in T_p^*M$, choose $\phi \in C^\infty(M)$, s.t. $\phi(p) = 0$, $d\phi(p) = \xi$ and choose $e \in E_{0,p}$, $s_0 \in C^\infty(M, E_0)$ s.t. $s_0(p) = e$. Then we have $\hat{D}\colon C^\infty(TM, \mathscr{L}(\pi^*(E_0), \pi^*(E_1)))$ for $\pi\colon TM \to M$ and

$$\hat{D}(\xi)\{e\} = \frac{i^l}{l!} D(\phi^l s_0)(p) \tag{I-37}$$

Alternatively we can write

$$= \lim_{t\to\infty} (t^{-l} e^{-it\phi(p)} D(e^{it\phi} s_0)(p)) \tag{I-38}$$

We call D *elliptic* iff $\hat{D}(\xi)$ is invertible for $\xi \neq 0$, i.e. $\hat{D}(\xi)$ is invertible on $T^*M \setminus \{0\}$. So to construct a parametrix we would want to form $\hat{D}(\xi)^{-1}$ in $\xi \neq 0$.

Pseudodifferential operators are linear operators A with domain $C^\infty(M, E_0)$ and image in $C^\infty(M, E_1)$. Each pseudodifferential operator has a symbol $\hat{A}$ and the *symbol map* Sym: $A \mapsto \hat{A}$ maps ΨDO into the *symbol space*

$$C^\infty(T^*M \setminus \{0\}, \mathscr{L}(\pi^* E_0, \pi^* E_1) \text{ positive homogeneous of degree } l \text{ for } l \in \mathbb{R}). \tag{I-39}$$

Then we propose the following axioms:

A1 The symbol map is surjective and if $\hat{A}_1 = \hat{A}_2$ of order l then $A_1 - A_2$ is of order at most $l - 1$.

A2 For $A_1 \in \Psi\mathrm{DO}_{l_1}(E_0, E_1)$, $A_2 \in \Psi\mathrm{DO}_{l_2}(E_1, E_2)$ we have $\widehat{A_2 A_1} = \hat{A}_1 \hat{A}_2$.

A3 $A \in \Psi\mathrm{DO}_l(E_0, E_1)$ extends to $\mathscr{L}(H_1, H_2)$ if $l \leq 0$, to $\mathcal{K}(H_1, H_2)$ if $l < 0$ and even to $\mathcal{K}_1(H_1, H_2)$ if $l < -m$.

A4 $A \in \Psi\mathrm{DO}_l(E_0, E_1)$ implies $A^* \in \Psi\mathrm{DO}_l(E_1, E_0)$

Now choose an operator $\Lambda_t, t \in \mathbb{R}$, positive definite, i.e. $\langle \Lambda_t s, s \rangle \geq 0$ and equal to zero only if $s = 0$. $\Lambda_t \in \Psi\mathrm{DO}_t(E, E)$ with $\hat{\Lambda}_t(\xi) := \langle \xi \rangle^t$.

64. **Definition: "Sobolev Space"**

We define the *Sobolev space* of order t for E

$$H^t(E) := \operatorname{dom} \Lambda_t \subset L^2(M, E) \tag{64.1}$$

Application to Elliptic Regularity (Calderón) Take $D \in \Psi\mathrm{DO}_l$, $l > 0$ with $\hat{D}_{pr}(x, \xi)$ invertible for $\xi \neq 0$. Then we want to show that $s \in \operatorname{dom} D \subset L^2, Ds \in C^\infty(M, E)$ implies $s \in C^\infty(M, E)$.

Construct $R_0 \in \Psi\mathrm{DO}_{-l}$ s.t. $\hat{R}_{0pr} = (\hat{D}_{pr})^{-1}$ outside the zero-section. Then

$$\widehat{R_0 D}_{pr} = \hat{R}_{0pr} \hat{D}_{pr} = I_{E_0} \tag{I-40}$$

$$\Rightarrow R_0 D = I + Q \qquad \text{with } Q \in \Psi\mathrm{DO}_{-1}(M, E_0) \subset \mathcal{K}(H_1, H_2) \tag{I-41}$$

Put for any $N \in \mathbb{N}$:

$$R_{N+1} := \sum_{k=0}^{N} (I - R_0 D)^k R_0 = R_0 \sum_{k=0}^{N} (I - D R_0)^k \tag{I-42}$$

$$\Rightarrow I - R_N D = (I - R_0 D)^N = Q^N \in \Psi\mathrm{DO}_{-N}(M, E_0) \tag{I-43}$$

(analogous for DR_0)

Hence for $N > m$, R_N is of trace class and thus a good parametrix.

65. **Theorem:**

Let $A \in \Psi\mathrm{DO}_l(M, E_0, E_1)$ be elliptic for $l > 0$. Then

1. $\bar{A}\colon H^l(M, E_0) \to H^0(M, E_1)$ and $\|s\|_{H^l(M,E_0)}$ is equivalent to the graph norm of $\bar{A}$.
2. $\bar{A}$ is a Fredholm operator, and

$$\operatorname{ind}\bar{A} = \operatorname{tr}_{H_1}(I_{H_1} - R_N A) - \operatorname{tr}_{H_2}(I_{H_2} - A R_N) \tag{65.1}$$

   if $N > m$.

3. *Elliptic regularity*: If $s \in H^l(M, E_0)$ and $\bar{A}s \in H^r(M, E_1)$, then $s \in H^k(M, E_0)$ where $k = \max\{l, l + r\}$.
4. $\overline{A^\dagger}\colon H^l(M, E_1) \to H^0(M, E_0)$ is actually equal to A^*.

Proof (65)

1. Is already known.
2. $\bar{A} \in \mathscr{F}_{cl}(H_1, H_2)$ by the parametrix construction, the index formula was already proved.
3. If $s \in H^l(M, E_0)$ then $\bar{A}s \in H^0 = L^2(M, E_1)$. Now assume $\bar{A}s \in H^r(M, E_0)$ for some $r > 0$. Then we apply R_N, N large

$$\bar{A}s =: s' \in H^r(M, E_0) \Rightarrow R_N \bar{A}s = R_N s' \in H^{r'+l}(M, E_0) \tag{65.2}$$

$$= \underbrace{\underbrace{(R_N\bar{A} - I_{H_1})}_{Q_0^N} s}_{\in H^N(M,E_0)} + s \Rightarrow s = R_N s' + (I_{H_1} - R_N\bar{A})s \in H^{\min\{r'+l,N\}}(M, E_0) \tag{65.3}$$

   In particular, $\bar{A}s \in C^\infty(M, E_1) \Rightarrow s \in C^\infty(M, E_0)$.

I.4.4. Elliptic Complexes

66. **Definition: "Differential Complex" and "Elliptic Complex"**

Let $E_i \to M$ be a finite sequence of vector bundles over a manifold M. Then a sequence of differential operators $D_i \in \mathrm{Diff}(M, E_i, E_{i+1})$ is called a *differential complex*. We write:

$$0 \longrightarrow E_0 \xrightarrow{D_0} E_1 \xrightarrow{D_1} \dots \xrightarrow{D_{k-1}} E_k \longrightarrow 0 \tag{66.1}$$

If the corresponding sequence of *principal symbols* $D_i \in C^\infty(T^*M, \mathscr{L}(\pi^*E_i, \pi^*E_{i+1}))$,

$$0 \longrightarrow E_0 \xrightarrow{\hat{D}_0(\xi)} E_1 \xrightarrow{\hat{D}_1(\xi)} \dots \xrightarrow{\hat{D}_{k-1}(\xi)} E_k \longrightarrow 0 \tag{66.2}$$

is *exact* outside the zero section, i.e. for $\xi \in T^*M \setminus M$, we call the differential complex *elliptic*.

Main example: The *de Rham complex* and its symbols are

$$0 \longrightarrow \lambda^0(M) \xrightarrow{d} \lambda^1(M) \xrightarrow{d} \dots \xrightarrow{d} \lambda^m(M) \longrightarrow 0 \tag{I-44}$$

$$0 \longrightarrow \Lambda^0 T^*M \xrightarrow{i\,\mathfrak{w}(\xi)_0} \Lambda^1 T^*M \xrightarrow{i\,\mathfrak{w}(\xi)_1} \dots \xrightarrow{i\,\mathfrak{w}(\xi)_{m-1}} \Lambda^m T^*M \longrightarrow 0 \tag{I-45}$$

Now for $\xi \neq 0$ we have $\ker \mathfrak{w}(\xi)_{i+1} = \operatorname{im} \mathfrak{w}(\xi)_i$, i.e. this sequence is *exact*, thus $(\lambda^j(M), d^j)$ is an *elliptic complex*. But notice that in general $H^j_{dR}(M) := \ker d^j / \operatorname{im} d^j \neq 0$.
Let M be compact as always, then

$$\begin{matrix} \overline{d^j}\colon H^1(M, \Lambda^j T^*M) & \longrightarrow & L^2(M, \Lambda^{j+1}T^*M) \\ & \searrow & \uparrow \\ & & \ker \overline{d^j} \subset \ker \overline{d^{j+1}} \subset H^1(M, \Lambda^{j+1}T^*M) \end{matrix} \tag{I-46}$$

P Problem: show $\operatorname{im} \overline{d^j} \subset \ker \overline{d^{j+1}}$.
Thus we define the *closed de Rham complex*: $(\lambda^j(M), \overline{d^j})$.

Note If $\varepsilon|_{\Lambda^j T^*M} = (-1)^j I_{\Lambda^j T^*M}$ then

$$\varepsilon_{j+1}\overline{d^j} + \overline{d^j}\varepsilon_j = 0 \tag{I-47}$$

67. **Lemma:**
Let $A \in \mathcal{C}(H)$ with adjoint $A^* \in \mathcal{C}(H)$ and $A^2 = 0$, i.e. $\operatorname{im} A \subset \ker A \subset \operatorname{dom} A$. Then

$$D := A + A^* \qquad \text{with } \operatorname{dom} D = \operatorname{dom} A \cap \operatorname{dom} A^* \tag{67.1}$$

dense in H is self-adjoint.

68. **Theorem:** HODGE **Decomposition Theorem**
Let H be any Hilbert space and $d \in \mathcal{C}(H)$ with $d^2 = 0$, then:

1. There is an orthogonal decomposition (called the *Kodaira-Hodge decomposition*)

$$\begin{aligned} H &= \ker d \cap \ker d^* \oplus \overline{\operatorname{im} d} \oplus \overline{\operatorname{im} d^*} \\ &= \underset{\textit{harmonic}}{H_h} \oplus \underset{\textit{closed}}{H_{cl}} \oplus \underset{\textit{coclosed}}{H_{ccl}} \end{aligned} \tag{68.1}$$

2. Define the *Hodge operator* $D := d + d^*$, then D is densely defined and self-adjoint.

3. If D is Fredholm then $\operatorname{im} d = \overline{\operatorname{im} d}$, $\operatorname{im} d^* = \overline{\operatorname{im} d}$ and we have the *Hodge decomposition* $H = H_h \oplus \operatorname{im} d \oplus \operatorname{im} d^*$ and $H_h = \ker D \simeq \ker d / \operatorname{im} d$ is finite dimensional.

Notation: $\ker d / \operatorname{im} d = \mathscr{H}(H, d)$ is called the *homology* of (H, d) and is usually a topological quantity.

4. D is Fredholm iff $\dim \ker d / \operatorname{im} d < \infty$.

5. If $\alpha \in \mathscr{L}(H)$ is a self-adjoint involution, i.e. $\alpha^{-1} = \alpha = \alpha^*$ and d is odd ($d\alpha + \alpha d = 0$) then if D is Fredholm we have

$$\operatorname{ind} D^+ = \dim \mathscr{H}^+(H, d) - \dim \mathscr{H}^-(H, d) \tag{68.2}$$

Here H can also be constructed from a non-compact manifold.

Proof (68)

1. We have a decomposition

$$H = \ker d \oplus (\ker d)^\perp = \ker d \oplus \overline{\operatorname{im} d^*} \tag{68.3}$$

$$= \overline{\operatorname{im} d} \oplus \ker d \cap \ker d^* \oplus \overline{\operatorname{im} d^*} \simeq (\ker d \cap \ker d^*) \oplus \overline{\operatorname{im} d} \oplus \overline{\operatorname{im} d^*} \tag{68.4}$$

Since D is Fredholm we know that $\operatorname{im} D = \left\{ dx + d^*x \;\middle|\; x \in \operatorname{dom} d \cap \operatorname{dom} d^* \right\}$ is closed and $\langle dx, d^*x \rangle = \langle d^2x, x \rangle = 0$.

$$H = \underbrace{\mathscr{H} \oplus \overline{\operatorname{im} d}}_{\subset \ker d} \oplus \underbrace{\overline{\operatorname{im} d^*}}_{\subset \operatorname{dom} d^*} \tag{68.5}$$

$$\operatorname{dom} d = \mathscr{H} \oplus \overline{\operatorname{im} d} \oplus \overline{\operatorname{im} d^*} \cap \operatorname{dom} d \tag{68.6}$$

$$\operatorname{dom} D = \mathscr{H} \oplus \overline{\operatorname{im} d} \cap \operatorname{dom} d^* \oplus \overline{\operatorname{im} d^*} \cap \operatorname{dom} d \ni x = x_h \oplus x_{cl} \oplus x_{ccl} \tag{68.7}$$

$$\Rightarrow dx + d^*x = d^*x_{cl} + dx_{ccl} \tag{68.8}$$

Where the indices stand for *harmonic*, *closed* and *coclosed*.

If $y_{ccl} = \lim_{n\to\infty} dx_{ccl,n}$ then also $y_{ccl} = \lim_{n\to\infty} Dx_{ccl,n} = D\bar{x}_{ccl} = d\bar{x}_{ccl}$.

2. Assume that D is Fredholm. Then $\ker d = H_h \oplus H_{cl}$, so $\operatorname{dom} d = H_h \oplus H_{cl} \oplus (H_{ccl} \cap \operatorname{dom} d)$ and $H_{ccl} \cap \operatorname{dom} d$ has to be dense in H_{ccl}. In the same way we get that $H_{cl} \cap \operatorname{dom} d^*$ is dense in H_{cl}. So then $\operatorname{dom} d \cap \operatorname{dom} d^* = H_h \oplus (H_{cl} \cap \operatorname{dom} d^*) \oplus (H_{ccl} \cap \operatorname{dom} d)$ is dense in H and D maps H_h to 0, $H_{cl} \cap \operatorname{dom} d^*$ as d^* to H_{ccl} and $H_{ccl} \cap \operatorname{dom} d$ like d to H_{cl}. Now $d \in C(H_{ccl}, H_{cl})$ with adjoint $d^* \in C(H_{cl}, H_{ccl})$.

Example Apply this to the *de Rham complex* on M compact, then

$$d^\pm \in \operatorname{Diff}_1(M, \Lambda^{ev/odd} T^*M, \Lambda^{odd/ev} T^*M) \tag{I-48}$$

$$\bar{d}^*, \bar{d} \in \operatorname{Diff}_1(M, H^1(M, \Lambda T^*M, \Lambda T^*M)) \tag{I-49}$$

$$\widehat{(d + d^\dagger)}(\xi) = i(\mathfrak{w}(\xi) - \iota(\xi^\#)) = ic(\xi) \tag{I-50}$$

$$-c(\xi)^2 = |\xi|^2 I_{\Lambda T^*M} \tag{I-51}$$

Here

$$\begin{aligned}\mathscr{H}(\lambda_{(2)}(M), d) &= \ker d / \operatorname{im} d = \bigoplus_{j\geq 0} \ker d^j / \operatorname{im} d^{j-1} \\ &= \bigoplus_{j\geq 0} \mathscr{H}^j_{dR}(M) \simeq \bigoplus_{j\geq 0} \mathscr{H}^j_{sing}(M, \mathbb{R})\end{aligned} \tag{I-52}$$

This gives us the following

69**. Corollary:**
$D^+ \in \mathscr{F}_{cl}(H^+, H^-)$ is also Fredholm with

$$\operatorname{ind} D^+ = \dim \mathscr{H}^+ - \dim \mathscr{H}^- = \chi(M) \tag{69.1}$$

if M is orientable.

Example Consider M^m, a closed, oriented manifold with a Riemannian metric g^{TM}. We introduce the *complex volume*

$$\omega_M := i^{[\frac{m+1}{2}]} c(e_1) \circ \ldots \circ c(e_m) \tag{I-53}$$

where (e_i) is a loonf for TM and c denotes the natural Clifford action on $\Lambda T^* M$: $c(e_j) = \mathfrak{w}(e^j) - \iota(e_j)$.
In physics ω is called the *chirality operator*.

P

70**. Lemma: Properties of ω**

1. $\omega^\dagger = \omega, \omega^2 = I_{\Lambda T^* M}$
2. $c(X)\omega = (-1)^{m+1} \omega c(X), X \in TM$
3. $\nabla^{LC} \omega = 0$
4. $d\omega = (-1)^{m+1} \omega d^\dagger$
   This implies $D\omega + (-1)^{m+1} \omega D = 0$ and $d^\dagger = (-1)^{m+1} \omega d \omega$
5. Assume m is even and express ω by the *Hodge star operator* $*$.

Then ω splits

$$\lambda_{(2)}(M) = \lambda^+_{(2)}(M) \oplus \lambda^-_{(2)}(M) \tag{I-54}$$

$$\mathscr{H}(\lambda_{(2)}(M), d) = \mathscr{H}^+ \oplus H^-(\lambda_{(2)}(M), d) \tag{I-55}$$

P

Then again

$$\operatorname{ind} D^+ = \dim \mathscr{H}^+ - \dim \mathscr{H}^- = \dim \mathscr{H}^{\frac{m}{2},+} - \dim \mathscr{H}^{\frac{m}{2},-} =: \textit{signature of } M \tag{I-56}$$

$$= \operatorname{tr} \beta \tag{I-57}$$

where

$$\beta(\eta) = \int_M \eta \wedge \omega\eta \tag{I-58}$$

P

Show that ind $D^* = 0$ if $m \in 2 + 4\mathbb{Z}$.

So an interesting signature arises only for $m \in 4\mathbb{Z}$. In that case the complex volume element is $\omega = (-1)^{m/4} c(e_1) \circ \ldots \circ c(e_m)$.

71**. Theorem:** HIRZEBRUCH

For M closed and oriented

$$\operatorname{sig} M = \operatorname{ind} D^+ = \int_M L(R^{LM}/2\pi i) \tag{71.1}$$

with the L-polynomial as defined in Definition 29 (Some Concrete Polynomials)

II. The Topological Index

II.1. Characteristic Classes

The important insight here is that *characteristic classes* (short cc) depend only on the vector bundle.

Recall: We have vector bundles E, E_0, E_1 over a manifold M and differential forms with values in a vector bundle:

$$\lambda(M, E) := C^\infty(M, E \otimes \Lambda T^* M) \tag{II-1}$$

$$\lambda^p(M, E) \ni \omega = \sum_j \sigma_j u^j =: \sigma\{u\}, \quad u^j \in \lambda^p(U), \tag{II-2}$$

where σ_j is a frame of E over U.

In this way we can take $\lambda^0(M, \mathscr{L}(E)) = C^\infty(M, \mathscr{L}(E))$ which gives a matrix of $C^\infty(U)$ elements for any section A over U and we can calculate

$$\sigma \cdot A \colon \sigma_k A^k_j = (\sigma \cdot A)_j \tag{II-3}$$

$$(\sigma \cdot A \cdot u) = \sigma_j (A^j_k u^k) = (\sigma_j A^j_k) u^k \tag{II-4}$$

$$\sigma' := \sigma A, \quad \sigma'\{u'\} = \sigma\{u\} = \sigma A\{u'\} \Leftrightarrow Au' = u \Leftrightarrow u' = A^{-1} u \tag{II-5}$$

where we use the Einstein summation convention. Then for forms $A \in \lambda^p(M, \mathscr{L}(E))$:

$$\Rightarrow A^j_k \in \lambda^p(U) \tag{II-6}$$

$$\Rightarrow \sigma A \wedge u = \sigma_j A^j_k \wedge u^k \tag{II-7}$$

$$A, B \in \lambda(M, \mathscr{L}(E)) \tag{II-8}$$

$$(A \wedge B)^j_k := A^l_k \wedge B^j_l \tag{II-9}$$

Now for E ungraded and $A \in \lambda^p(M, \mathscr{L}(E))$ and $B \in \lambda^q(M, \mathscr{L}(E))$ the *supercommutator* is

$$[A, B]_s = A \wedge B - (-1)^{pq} B \wedge A \tag{II-10}$$

If we change frames by $\sigma' = \sigma F$ we get

$$\sigma' A' u' = \sigma A u \Leftrightarrow u' = Fu, A' = F^{-1} A F \tag{II-11}$$

Now take a connection ∇ on $\lambda(M, E)$, i.e. for $s \in \lambda(M, E), \omega \in \lambda^p(M)$:

$$\nabla(s\omega) = (\nabla s)\omega + s(-1)^{|\omega|} d\omega \tag{II-12}$$

$$\nabla \sigma_j = \sigma_k \Gamma^k_j \in \lambda^1(U, E) \tag{II-13}$$

where (σ_j) is again a frame. Now with $\sigma\{u\} \in C^\infty(U, E)$

$$\nabla \sigma\{u\} = \nabla(\sigma_j u^j) = (\nabla \sigma_j) u^j + \sigma_j du^j \tag{II-14}$$

$$= \sigma_k \Gamma^k_j u^j + \sigma_k du^k = \sigma\{d\{u\} + \Gamma\{u\}\} \tag{II-15}$$

P

Then for a change of frame $\sigma' = \sigma F$ we get the formula

$$\Gamma' = F^{-1}\Gamma F + F^{-1}dF \tag{II-16}$$

Take $A, B \in \lambda(M, \mathscr{L}(E))$, then

$$\nabla(A \wedge B) = \nabla A \wedge B + (-1)^{|A|} A \wedge \nabla B \tag{II-17}$$

and for $A \in \lambda(M, \mathscr{L}(E))$ and $u \in \lambda(M, E)$ we get

$$\nabla(A \cdot u) = (\nabla A)u - (-1)^{|A|} A \cdot \nabla u \tag{II-18}$$

P

Show that Equation II-18 translates locally to

$$\{\nabla A\} = d\{A\} + [\Gamma, \{A\}]_s \tag{II-19}$$

72. **Definition: "Trace of Forms"**
For $A \in \lambda(M, \mathscr{L}(E))$ we define the *trace* as

$$\operatorname{tr} A := \sum_j A^j_j \in \lambda(M) \tag{72.1}$$

Then tr A is well defined and

$$d \operatorname{tr} A = \operatorname{tr} \nabla A \tag{72.2}$$

for any connection ∇ on E.

Curvature $= \nabla^2 \in \lambda^2(M, \mathscr{L}(E))$
For $u \in \lambda(M, E)$ we get

$$\nabla^2 u = \nabla(\nabla u) = \sigma\{d\{\nabla u\} + \Gamma\{\nabla u\}\} = \sigma\{d\{d\{u\} + \Gamma\{u\}\} + \Gamma\{d\{u\} + \Gamma\{u\}\}\} \tag{II-20}$$

$$= \sigma\{d(\Gamma\{u\}) + \Gamma \wedge d\{u\} + \Gamma \wedge \Gamma\{u\}\} = \sigma\{d\Gamma\{u\} + \Gamma \wedge \Gamma\{u\}\} \tag{II-21}$$

$$\Rightarrow \{\nabla^2\} = d\Gamma + \Gamma \wedge \Gamma = d\Gamma + \frac{1}{2}[\Gamma, \Gamma] \tag{II-22}$$

$$(\nabla^2)^k_j = d\Gamma^k_j + \Gamma^l_j \wedge \Gamma^k_l \tag{II-23}$$

P

So again for $A \in \lambda(M, \mathscr{L}(E))$ we get

$$\{\nabla^2 A\} = [\{\nabla^2\}, \{A\}]_s \tag{II-24}$$

73. **Theorem:** BIANCHI **Identity**
The curvature tensor is constant:

$$\nabla(\nabla^2) = 0 \tag{73.1}$$

Proof (73)

$$\nabla^3 u = \nabla(\nabla^2 u) = (\nabla(\nabla^2))u + \nabla^2(\nabla u) \tag{73.2}$$
$$= \nabla^2(\nabla u) \tag{73.3}$$
$$\Rightarrow \nabla(\nabla^2) = 0 \tag{73.4}$$

Variation of Connections and Curvature Take two connections ∇^0, ∇^1 on E, then

$$\nabla^t := (1-t)\nabla^0 + t\nabla^1, t \in [0,1] \tag{II-25}$$

is a connection and

$$\nabla^t - \nabla^0 = t(\nabla^1 - \nabla^0) \stackrel{\{\cdot\}}{=} t(\Gamma^1 - \Gamma^0) =: t\Theta^{0,1} \in \lambda^1(M, \mathscr{L}(E)) \tag{II-26}$$

because $\Theta^{0,1}$ transforms nicely under a change of frame (compare Equation II-16). So then

$$\frac{d}{dt}\nabla^t =: \dot{\nabla}^t = \Theta^{0,1} \tag{II-27}$$
$$\frac{d}{dt}\nabla^{t,2} = (\nabla^t\nabla^t)^{\cdot} = \dot{\nabla}^t\nabla^t + \nabla^t\dot{\nabla}^t = \Theta^{0,1}\nabla^t + \nabla^t\Theta^{0,1} \tag{II-28}$$
$$\Rightarrow \dot{\nabla}^{t,2}u = \Theta^{0,1} \wedge \nabla^t u + \nabla^t(\Theta^{0,1}u) \tag{II-29}$$
$$= \Theta^{0,1} \wedge \nabla^t u + (\nabla^t\Theta^{0,1}) \wedge u + (-1)\Theta^{0,1} \wedge \nabla^t u = (\nabla^t\Theta^{0,1}) \wedge u \tag{II-30}$$

Now we introduce

$$\Omega := -\frac{1}{2\pi i}\nabla^2 \tag{II-31}$$

74. **Definition: “Adams Form” and “Chern Form”**

We define the *Adams forms* as

$$\psi_k := \operatorname{tr} \Omega^k \in \lambda^{2k}(M), k \in \mathbb{Z}_+ \tag{74.1}$$

and the *Chern forms* as

$$\det(I - \lambda\Omega) = \sum_{j\geq 0} \lambda^j c_j, \quad c_j \in \lambda^{2j}(M) \tag{74.2}$$

We should denote the dependencies by writing $\psi_k(E, \nabla^E)$.

Assume M^m is a closed and oriented manifold. Then the following questions arise:

1. Is $d\psi_k = dc_k = 0$?
2. Do ψ_k and c_k depend on ∇?

75. **Lemma:**
The *Adams forms* are closed.

Proof (75) Note that λ^{ev} commutes with any form. Then

$$d\psi_k = d \operatorname{tr} \Omega^k \overset{72}{=} \operatorname{tr}(\nabla\Omega^k) \overset{!}{=} \operatorname{tr}(k\Omega^{k-1}\nabla(\nabla^2)(-\frac{1}{2\pi i})) \overset{73}{=} 0 \tag{75.1}$$

76. **Corollary:**
Since any c_k is a linear combination of some ψ_k, all c_k are closed.

P

Proof (76) Show this by using NEWTON's identities.

77. **Corollary:**
Let f be holomorphic in a neighbourhood of 0, then $f(\Omega)$ is closed.

78. **Theorem:**

P

Any $f(\Omega)$ is independent of the choice of ∇, modulo d-boundaries.

$$f \in \mathbb{C}\{\{z\}\}, \quad \Omega = -\frac{1}{2\pi i}\nabla^{E,2} \in \lambda^2(M, \mathscr{L}(E)) \tag{II-32}$$

$$E \to M, \nabla^E \tag{II-33}$$

$$\Rightarrow f_+(E) := \operatorname{tr}_E\{f(\Omega) = \sum_{j\geq 0} f_j\Omega^j\} = \sum_{j\geq 0} f_j \underbrace{\psi_j(E)}_{\in\lambda^{2j}(M)} \in \lambda^{ev}(M) \tag{II-34}$$

Then

$$f_+(E), f_\cdot(E) := \det f(\Omega) \in \lambda^{ev}(M) \tag{II-35}$$

are closed and their *de Rham class*, which we then call *characteristic class*, does not depend on the choice of ∇^E.

79. **Lemma: Functorial Properties of Characteristic Classes**
We have $f(E) = f_{+/\cdot}(E) \in H^{\cdot}_{dR}(M)$. Now $\phi \in C^\infty(M, N)$ induces a homomorphism $\phi^* = H^{\cdot ev}_{dR}(N) \to H^{\cdot ev}_{dR}(M)$ of commutative rings. Then

1. $\phi^*(f(E)) = f(\phi^* E)$
2. $f(E^*) = \check{f}(E)$ where $\check{f}(z) = f(-z)$
3. $f_+(E_0 \oplus E_1) = f_+(E_0) + f_+(E_1)$
4. $f_\cdot(E_0 \otimes E_1) = f_\cdot(E_0) \wedge f_\cdot(E_1)$

As in Definition 29 (Some Concrete Polynomials) we have

80**. Theorem: Most Frequent Characteristic Classes**

1. $\psi^k(E) = \operatorname{tr} \Omega^k$ are the *Adams classes* (mostly technical)
2. $c_k(E)$ are the *Chern classes* (from $\det(I - \lambda\Omega)$)

$$c(E) = \sum_{j\geq 0} c_k(E) \tag{80.1}$$

3. $\operatorname{ch} E = \operatorname{tr} e^{\Omega}$ the *Chern character*
4. $\hat{A}(E) = \det\left(\frac{\Omega}{\sinh \Omega}\right)^{\frac{1}{2}}$ the $\hat{A}$*-genus*
5. $\operatorname{Td}(E) = \det\left(\frac{\Omega}{I-e^{-\Omega}}\right)$ the *Todd class* (after TODD)

P **Use in Topology** If E is trivial then all characteristic forms are zero and they are equal for isomorphic bundles.

81**. Lemma: Properties of the Chern Character**

Let E_0, E_1 be vector bundles over M, then we have

Additivity: $\operatorname{ch}(E_0 \oplus E_1) = \operatorname{ch}(E_0) + \operatorname{ch}(E_1)$

Multiplicity: $\operatorname{ch}(E_0 \otimes E_1) = \operatorname{ch}(E_0) \wedge \operatorname{ch}(E_1)$

Proof (81) of the multiplicity:

Take connections ∇^0, ∇^1 on E_0, E_1. Then there is a connection $\nabla^{E_0\otimes E_1}$ on $E_0 \otimes E_1$ manufactured from

$$\nabla^{E_0\otimes E_1}(s_0 \otimes s_1) = (\nabla^{E_0} s_0) \otimes s_1 + s_0 \otimes (\nabla^{E_1} s_1) \tag{81.1}$$

Compute Γ locally:

$$\sigma_{0i} \otimes \sigma_{1j} | \{\nabla^{E_0\otimes E_1} u\} = d\{u\} + \Gamma \wedge \{u\} \tag{81.2}$$

$$\Rightarrow \nabla^{(E_0\otimes E_1),2} = d\Gamma + \Gamma \wedge \Gamma \tag{81.3}$$

$$= \nabla^{E_0,2} \otimes I_1 + I_0 \otimes \nabla^{E_1,2} \tag{81.4}$$

$$\operatorname{ch}(E_0 \otimes E_1) = \operatorname{tr} e^{\nabla^{0,2}\otimes I_1 + I_0 \otimes \nabla^{1,2}} = \operatorname{tr}(e^{\nabla^{0,2}\otimes I_1} e^{I_0\otimes\nabla^{1,2}}) = \operatorname{tr}_{E_0} e^{\nabla^{0,2}} \operatorname{tr}_{E_1} e^{\nabla^{1,2}} \tag{81.5}$$

82**. Theorem:**

Take a vector bundle $E \to M^m$, $m \in 2\mathbb{Z}$, with connection ∇^E and form $(\Lambda^k E)_{k\geq 0}$ with the natural even/odd grading. Then

$$\operatorname{ch} \Lambda^{ev} E^* - \operatorname{ch} \Lambda^{odd} E^* = c_m(E)(\operatorname{Td})^{-1}(M) \tag{82.1}$$

Remark on Real Vector Bundles Formally we can repeat what we did before, but f needs to be real. Then we have again

$$\det(I - \lambda\nabla^{E,2}) = \sum_{j\geq 0} \lambda_j^c(E) \tag{II-36}$$

The relation $f(E^*) = \check{f}(E)$ with $\check{f}(z) = -z$.

$$\Rightarrow f(E) = \check{f}(E) \Leftrightarrow f_{2j+1} = 0 \forall j \in \mathbb{Z}_+ \tag{II-37}$$

$$\Rightarrow c_{2j+1}(E) = 0 \forall j \tag{II-38}$$

83. **Definition: "Pontrjagin Form"**

If E is real, then we define

$$p_j(E) := c_{2j}(E) \in H^{4j}_{dR}(M) \tag{83.1}$$

as the j-th *Pontrjagin form*

Let $E \to M^m$, $m \in 2\mathbb{Z}$ be real and oriented. Choose a metric g^E, a metric connection ∇^E and a loonf (e_j), then $\nabla^{E,2}$ is skew-symmetric.

Now we have a fibre $V = E_p$ as an Euclidean vector space and thereon the skew-symmetric operators denoted by $A \in \mathscr{L}_{as}(V)$. Then there is an isomorphism

$$\mathscr{L}_{as}(V) \ni A \mapsto \sum_{j<k} \langle Ae_j, e_k\rangle e^j \wedge e^k \in \Lambda^2(V) \tag{II-39}$$

and the other way around

$$\tilde{A} \mapsto e_\wedge^{\tilde{A}} \mapsto \int^B e_\wedge^{\tilde{A}} \tag{II-40}$$

Here we call

$$\int^B e_\wedge^{\tilde{A}} =: \mathrm{Pf}(-A) \in \Lambda_p^m V \tag{II-41}$$

the *Euler form* of E and under a change of orientation we have

$$\mathrm{Pf}(E_{or}) = -\mathrm{Pf}(E_{-or}) \tag{II-42}$$

Thus the following is independent of the orientation:

$$\int_M \mathrm{Pf}(E) =: \chi(E) \tag{II-43}$$

P **Interesting Fact:** $\mathrm{Pf}(A)^2 = \det(A)$

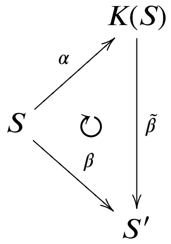
II.2. Some Elements of K-Theory

> "K-Theory is the linear algebra of manifolds."

For a reference see the classic book [Ati94].

Take a locally compact Hausdorff space X. Then we define $\mathrm{Vect}(X)$ to be the isomorphism classes of continuous $\mathbb{C}$-vector bundles on X. If X is just one point, $X = *$, we have $\mathrm{Vect}(*) = \mathbb{Z}_+ \subset \mathbb{Z}$. In general $\mathrm{Vect}(X)$ is a commutative semiring for every X. To get nicer objects we use the

Grothendieck Construction Let S be an abelian semi-group with 0. Then we want an abelian group $K(S)$ admitting a semigroup homomorphism $\alpha\colon S \to K(S)$ and satisfying the following universal property for any other group S' and any semigroup homomorphism β:



$$\tilde{\beta} \circ \alpha = \beta$$

84**. Definition: "Grothendieck Group"**

Let S be any semigroup. Take $F(S)$ as the free abelian group generated by S, $F(S) \simeq \mathbb{Z}^S$, with group homomorphism $S \ni s \to \langle s \rangle \in F(S)$. Then we take $E(S)$ as the subgroup of $F(S)$ generated by $(\langle s_1 + s_2 \rangle - \langle s_1 \rangle - \langle s_2 \rangle)$ and put the *Grothendieck group* as $K(S) \coloneqq F(S)/E(S)$.

We also introduce another construction. Let $\Delta S \coloneqq \{ (s, s) \mid s \in S \}$ and define

$$\mathscr{K}(S) \coloneqq {}^{S \times S}\!\big/_{\Delta S} \tag{84.1}$$

where $S \times S$ carries the usual product group structure.

85**. Theorem: Properties of the Grothendieck Group**

P

For a semigroup S and the above constructions we get:

1. $\mathscr{K}(S)$ is abelian and satisfies the universal property with the natural group homomorphism $S \to \mathscr{K}(S)$ in the form $\alpha(s) = (s, 0) + \Delta S$. So then $\mathscr{K}(S) \simeq K(S)$. We will thus only write $K(S)$ denoting any isomorphic group that satisfies said universal property.

2. Any element of $K(S)$ takes the form $\alpha(s_1) - \alpha(s_2)$, $s_1, s_2 \in S$.

3. α is injective iff S has the *cancellation property*:

$$s_1 + s = s_2 + s \Rightarrow s_1 = s_2 \tag{85.1}$$

for all $s, s_1, s_2 \in S$.

4. If S carries a semi-ring structure, it induces one on $K(S)$.
5. $\operatorname{Vect}(X)$ does not have the cancellation property.

Eilenberg-Steenrod Axioms The *Eilenberg-Steenrod axioms* describe properties that a reasonable *homology theory* should satisfy. First of all there has to be a functor which assigns each pair of topological spaces (X, A) with $A \subset X$ a $\mathbb{Z}$-graded abelian group $H(X, A) = (H_q(X, A))_{q\in\mathbb{Z}}$, called the *homology groups*. Here we write $(X, \varnothing) =: X$. Also for each continuous map $f: (X, A) \to (Y, B)$, i.e. $f: X \to Y$ and $f|_A : A \to B$, there is an induced group homomorphism $f_*: H(X, A) \to H(Y, B)$ which respects the grading. Given the natural injections $i: A \to X$ and $j: X \to (X, A)$ we get a short sequence:

$$H(A) \xrightarrow{i_*} H(X) \xrightarrow{j_*} H(X, A) \tag{II-44}$$

Then there is connecting *boundary operator* $\partial: H_*(X, A) \to H_{*-1}(A)$ (compare also the *snake lemma*).

Now the axioms are the following:

A1 $f = \mathrm{id}_{(X,A)} \Rightarrow f_* = \mathrm{id}_{H(X,A)}$

A2 The functor is covariant: $(gf)_* = g_* f_*$.

A3 The *boundary operator* is natural, i.e. the following diagram commutes:

$$\begin{array}{ccc} H_*(X, A) & \xrightarrow{\partial} & H_{*-1}(A) \\ \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle f_*} \\ H_*(Y, B) & \xrightarrow{\partial} & H_{*-1}(B) \end{array} \tag{II-45}$$

A4 The long sequence constructed from i, j and ∂ is exact:

$$\cdots \dashrightarrow H_{q+1}(X, A) \xrightarrow{i_*} H_{q+1}(X) \xrightarrow{j_*} H_{q+1}(A) \xrightarrow{\partial} H_q(X, A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(A) \dashrightarrow \cdots \tag{II-46}$$

A5 Homotopic maps f, g satisfy $f_* = g_*$.

A6 *Excision property*: Given (X, A) and U, V open s.t. $U \subset A$, $\bar{U} \subset V \subset A$ then

$$H(X \setminus U, A \setminus U) = H(X, A) \tag{II-47}$$

A7 *Dimension axiom*: $H_q(\mathrm{pt}) = 0$ for all $q \in \mathbb{Z} \setminus \{0\}$.

Cohomology For a *cohomology* theory there are similar axioms. We have again a functor, which now assigns the *cohomology classes*, denoted by $H^q(X, A)$. But now the functor is contravariant, i.e. for $f\colon (X, A) \to (Y, B)$ we get $f^*\colon H^*(Y, B) \to H^*(X, A)$ and instead of (A2) we have $(gf)^* = f^* g^*$. Also the boundary operator is replaced by a *coboundary operator* $\partial^*\colon H^*(A) \to H^{*+1}(X, A)$ which raises the degree. So in (A3) and (A4) we have to reverse the direction of all arrows. (A1), (A5), (A6) and (A7) stay the same.

K-Theory To build a cohomology theory out of K we restrict ourself for now to compact Hausdorff spaces X. We write $K(X) := K(\mathrm{Vect}(X))$ and use the following notation: $\mathrm{Vect}\, X \ni E \mapsto [E] \in K(X)$ for the projection onto $K(X)$. For $f\colon X \to Y$ we take the pullback of vector bundles $f^*\colon K(Y) \to K(X) =: K^0(X)$. As in Theorem 85 (Properties of the Grothendieck Group), $K(X)$ carries a ring structure given by the direct sum $\oplus$ and the tensor product $\otimes$ of vector bundles.

86**. Definition: "Reduced K-Group"**

Given a compact topological space X with marked point ∞ we denote the injection $\infty \to X$ by ∞ as well. Then the *reduced K-group* is

$$\tilde{K}(X, \infty) := \ker \infty^* \subset K(X) \tag{86.1}$$

If the basepoint is not relevant we may write $\tilde{K}(X)$.

87**. Definition: "First K-Cohomology Group"**

For a pair (X, A) with $A \subset X$ closed we set X/A as the space constructed from X by identifying all points of A. Then

$$K^0(X, A) := \tilde{K}(X/A, [A]) \tag{87.1}$$

In the case $A = \emptyset$ we have to make an extra effort because $[A]$ does not give a basepoint for X/A. Thus we choose some new $\infty \notin X$ and define $(X/\emptyset) := (X \cup \{\infty\})$. Then

$$K^0(X) := \tilde{K}(X \cup \{\infty\}, \{\infty\}) = \ker \infty^* = \left\{ [E] - [F] \,\middle|\, \mathrm{rk}\, E|_\infty = \mathrm{rk}\, F|_\infty \right\} = K(X) \tag{87.2}$$

After defining a suitable cohomology class K^1 there is an exact sequence

$$\begin{array}{ccccc}
K^0(X,A) & \xrightarrow{j^*} & K^0(X) & \xrightarrow{i^*} & K^0(A) \\
\uparrow{\scriptstyle \partial^*} & & \text{Bott periodicity} & & \downarrow{\scriptstyle \partial^*} \\
K^1(A) & \xleftarrow[i^*]{} & K^1(X) & \xleftarrow[j^*]{} & K^1(X,A)
\end{array} \tag{II-48}$$

A corresponding K-homology is possible but rather complicated.
Furthermore there is a *KK-theory* by KASPAROV.

88. **Definition: "K-Theory with Compact Support"**

If X is a locally compact Hausdorff space, we define

$$K(X) := \tilde{K}(\dot{X}, \{\infty\}) \tag{88.1}$$

where $\dot{X} := X \sqcup \{\infty\}$ is the *one-point (*ALEXANDROV*) compactification* of X with infinite point ∞. A neighbourhood basis of ∞ is given by

$$\mathcal{U}_\infty = \{\ X \setminus K \mid K \subset X \text{ compact }\ \} \tag{88.2}$$

Thus if X is compact, ∞ is an isolated point in $\dot{X}$ and, like in Definition 86 (Reduced K-Group), we get that the notation $K(X)$ is compatible with Definition 84 (GROTHENDIECK Group) where $S = \mathrm{Vect}(X)$.

For a manifold M we write

$$K_{cpt}(M) := K(T^*M) \tag{88.3}$$

89. **Lemma: Restriction Map**

For a locally compact Hausdorff space X and an open subset $U \subset X$ we define a projection map under identification of the points $\dot{X} \setminus U$:

$$P\colon \dot{X} \to \dot{X} \big/ \dot{X} \setminus U \simeq \dot{U} \tag{89.1}$$

Then there is an induced ring homomorphism $P^*\colon K(U) \to K(X)$.

II.3. K-Theory and Bundle Complexes

In the following let X be a locally compact Hausdorff space. We will now derive a construction of $K(X)$ in terms of *bundle complexes*.

90. **Definition: "Bundle Complex"**

A finite sequence $E_i \to X$ of continuous vector bundles together with a sequence of vector bundle morphisms $a_i \in C(X, \mathscr{L}(E_i, E_{i+1}))$ forms a *chain complex* of vector bundles if $\operatorname{im} a_i \subset \ker a_{i+1}$, which we simply call a *bundle complex*.

$$(E, \alpha)\colon 0 \longrightarrow E^0 \xrightarrow{\alpha_0} \dots \xrightarrow{\alpha_{n-1}} E^n \longrightarrow 0 \tag{90.1}$$

Then over each point $x \in X$ this yields a complex of vector spaces (E_x, α_x) and we define

$$\operatorname{supp}(E, \alpha) := \left\{\ x \in X \ \middle|\ (E_x, \alpha_x) \text{ is not exact }\ \right\}. \tag{90.2}$$

So when the *homology* of the complex is

$$\mathscr{H}(E) := \ker \alpha \big/ \operatorname{im} \alpha, \tag{90.3}$$

the *support* is the domain where the homology is non-trivial. We are only interested in complexes with compact support and denote them by $\mathscr{L}(X)$ and $\mathscr{L}^n(X)$ when the length is fixed.

91. Remark: Special Case

In particular any *differential complex* over a manifold M gives through its principal symbols a *bundle complex* over T^*M. If M is compact and the differential complex is elliptic, it has compact support.

92. Definition: "Homotopic Complexes"

We say two complexes $(E,\alpha),(F,\beta) \in \mathscr{L}(X)$ are equivalent $(E,\alpha) \sim (F,\beta)$ iff there is a continuous homotopy $(G,\gamma) \in \mathscr{L}(X \times [0,1])$ such that for $i_t\colon x \mapsto (x,t)$ we have

$$i_0^*(G,\gamma) = (E,\alpha) \qquad i_1^*(G,\gamma) = (F,\beta) \tag{92.1}$$

Then the bundles E^i and F^i have pairwise the same rank. We set

$$\mathscr{C}(X) := \mathscr{L}(X) \big/ \sim \tag{92.2}$$

93. Lemma:

P Isomorphic complexes are automatically homotopic and thus represent the same class in $\mathscr{C}$.

94. Remark: Representatives of Bundle Complexes

1. Any two complexes (E,α) and (F,β) that only differ on some compact set $L \subset X$, are homotopic, through $\gamma = t\alpha + (1-t)\beta$, because $\gamma = \alpha = \beta$ on $X \setminus L$.

2. Given a complex (E,α) with α defined only outside the compact set $L \subset X$ we take $\phi \in C_c(X)$ such that $\phi = 1$ in a neighbourhood of L and consider $(E,\tilde{\alpha})$ with

$$\tilde{\alpha}(x) = \begin{cases} 0, & \text{if } x \in L \text{ and} \\ (1-\phi(x))\alpha(x), & \text{otherwise} \end{cases} \tag{94.1}$$

   to get an element of $\mathscr{C}(X)$. This does not depend on the choice of ϕ, because of (1).

3. Because ranks are equal for homotopic complexes, their lengths must also be equal. Thus we may write $\mathscr{C}^n(X)$ for $\mathscr{L}_n(X)/\sim$. But if we now divide out the (up to homotopy) everywhere exact complexes $\mathscr{C}_\varnothing(X)$, we find that any element $(E,\alpha) \in \mathscr{C}(X)/\mathscr{C}_\varnothing(X)$ can be represented as a complex of length 2:

$$(\tilde{E},\tilde{\alpha})\colon 0 \longrightarrow E\big/\ker\alpha \longrightarrow \ker\alpha \longrightarrow 0 \tag{94.2}$$

   where $\tilde{E} = \bigoplus_{j=0}^n E^j$, $\tilde{\alpha} = \bigoplus_{j=0}^n \alpha_j \in \mathscr{L}(E)$ and $\tilde{\alpha}^2 = 0$. That means the complexes (E,α) and $(\tilde{E},\tilde{\alpha})$ have the same support and define the same homotopy class up to an exact
P complex. We also write $(\tilde{E}^0,\tilde{E}^1,\tilde{\alpha})$ for those complexes of length 2.

95. Lemma:

$\mathscr{C}(X)/\mathscr{C}_\varnothing(X)$ exhibits a commutative ring structure induced by the direct sum $\oplus$ and tensor product $\otimes$ on $\mathrm{Vect}(X)$.

Proof (95) Because of Remark 94 item 3 we can restrict ourselves to complexes of length 2.

- The usual direct sum of vector bundles induces a well-defined sum on $\mathscr{C}(X)/\mathscr{C}_\varnothing(X)$:

$$(E,\alpha)+(F,\beta) := (E\oplus F, \alpha\oplus\beta) \tag{95.1}$$

$$0 \longrightarrow E^0_v \oplus E^0_v \xrightarrow{\alpha_v\oplus\beta_v} E^1_v \oplus F^1_v \longrightarrow 0 \tag{95.2}$$

  is exact if $v \notin \operatorname{supp}(E,\alpha) \cup \operatorname{supp}(F,\alpha)$, which is compact. The neutral element is represented by $\mathscr{C}_\varnothing(X)$. To find an inverse for (E^0, E^1, α), choose a metric and consider the dual complex (E^1, E^0, α^*). The sum $(E^0, E^1, \alpha) \oplus (E^1, E^0, \alpha^*)$ is homotopic to

$$0 \longrightarrow E^0 \oplus E^1 \xrightarrow{\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} E^0 \oplus E^1 \longrightarrow 0 \tag{95.3}$$

  We cut it off outside the support: $\tilde\alpha := (1-\phi)\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}$ with $\phi \in C(X,\mathbb{R})$ and $\phi|_{\operatorname{supp}\alpha} = 1$. Then $\tilde\alpha$ is self-adjoint and invertible outside the support of α. Let $\operatorname{spec}\tilde\alpha_v = (\lambda_1,\ldots,\lambda_k)(v)$, $k = \dim E^0 = \dim E^1$. λ_j depends continuously on v, so $\#\{\lambda_j > 0\}$ is constant and $P_<, P_>$ are projections of constant rank. Thus with $\tilde E := E^0 \oplus E^1$ we have $(\tilde E, \tilde\alpha) = (\tilde E_>, \tilde\alpha_>) \oplus (\tilde E_<, \tilde\alpha_<)$. This is connected to the trivial complex $(\tilde E_>, \mathrm{id}) \oplus (\tilde E_<, -\mathrm{id})$ by the homotopies

$$\tilde\alpha_{>,v}(t) := (1-t)\tilde\alpha_{>,v} + t\,\mathrm{id}_{E_>} \tag{95.4}$$

$$\tilde\alpha_{<,v}(t) := (1-t)\tilde\alpha_{<,v} - t\,\mathrm{id}_{E_<} \tag{95.5}$$

  where $t \in [0,1]$. As we are dealing with complex vector bundles, $-\mathrm{id}$ is homotopic to id. Thus we have shown that $(\tilde E, \tilde\alpha)$ is homotopic to the trivial complex $(E^0 \oplus E^1, \mathrm{id})$ and thus there are indeed inverse elements wrt $\oplus$.

- To construct a product we write $(E,\alpha) * (F,\beta)$ for:

$$0 \longrightarrow E^0 \otimes F^0 \xrightarrow{\phi} E^0 \otimes F^1 \oplus E^1 \otimes F^0 \xrightarrow{\psi} E^1 \otimes F^1 \longrightarrow 0 \tag{95.6}$$

$$\phi = \mathrm{id}_{E^0} \otimes \beta + \alpha \otimes \mathrm{id}_{E^1}, \quad \psi = \alpha \otimes \mathrm{id}_{F^1} - \mathrm{id}_{E^1} \otimes \beta \tag{95.7}$$

$$\rightsquigarrow\ 0 \longrightarrow E^0 \otimes F^0 \oplus E^1 \otimes F^1 \xrightarrow{\phi+\psi^*} E^0 \otimes F^1 \oplus E^1 \otimes F^0 \longrightarrow 0 \tag{95.8}$$

  Here ϕ and ψ are injective.

To see how this tensor product arises, we need to look at the following notions.

96. **Definition: "Super Complex"**

A *super complex* (E,α,ε) over X is a *bundle complex* (E,α) together with a $\mathbb{Z}_2$-grading $\varepsilon \in C(X, \mathscr{L}(E))$ such that α is odd, i.e.

$$\varepsilon\alpha + \alpha\varepsilon = 0. \tag{96.1}$$

In particular ε should respect the $\mathbb{Z}$ grading of E, $\varepsilon(E^i) = E^i$, and be an involution, $\varepsilon^2 = \mathrm{id}_E$.

97**. Lemma: Representation of Super Complexes**

Any super complex (E, α, ε) is the sum of two super complexes $(E_1, \alpha_1, \varepsilon_1)$ and $(E_2, \alpha_2, \varepsilon_2)$ with ε_1 and ε_2 induced by the usual $\mathbb{Z}$-grading. That means

$$\varepsilon_i\big|_{E_i^j} = \begin{cases} \mathrm{id}_{E_i^j} & j \in 2\mathbb{Z} \\ -\mathrm{id}_{E_i^j} & j \notin 2\mathbb{Z} \end{cases} \quad \text{for } i = 1, 2. \tag{97.1}$$

We will call these *alternating super bundles*.

98**. Remark: Canonical Tensor Product**

Given two alternating super bundles of length 1, $(E_i, \alpha_i, \varepsilon_i)$ with $i = 1, 2$, the induced $\mathbb{Z}_2$-grading $\varepsilon := \varepsilon_1 \otimes \varepsilon_2$ on $E := E_1 \otimes E_2 = (E_1^0 \oplus E_1^1) \otimes (E_2^0 \oplus E_1^1)$ decomposes this space into the parts $E_1^0 \otimes E_2^0 \oplus E_1^1 \otimes E_2^1$ and $E_1^0 \otimes E_2^1 \oplus E_1^1 \otimes E_2^0$. So the resulting alternating bundle (E, α, ε) takes the form

$$0 \longrightarrow E_1^0 \otimes E_2^0 \oplus E_1^1 \otimes E_2^1 \longrightarrow E_1^0 \otimes E_2^1 \oplus E_1^1 \otimes E_2^0 \longrightarrow 0\,, \tag{98.1}$$

where we can define

$$\alpha := \alpha_1 \otimes \mathrm{id}_{E_2} + \varepsilon_1 \otimes \alpha_2. \tag{98.2}$$

99**. Theorem:**

Writing $\mathscr{C}^n_\varnothing(X)$ for everywhere exact complexes in $\mathscr{C}^n(X)$, we have

$$K(X) \quad \simeq \quad \mathscr{C}(X)\big/\mathscr{C}_\varnothing(X) \quad \simeq \quad \mathscr{C}^n(X)\big/\mathscr{C}^n_\varnothing(X) \tag{99.1}$$

for any $n \in \mathbb{N}$. Thus we will mostly write $K(X)$ for the latter construction too.

100**. Remark:**

P

1. If X is compact, then the isomorphism from $\mathscr{C}(X)/\mathscr{C}_\varnothing$ to $K(X)$ has the form

$$\mathscr{C}(X) \ni E \mapsto \sum_{j=0}^{n} (-1)^j [E_j]_{K(X)} = \chi(E) \tag{100.1}$$

2. If in Definition 88 (K-Theory with Compact Support) X is not compact, i.e. ∞ is not isolated, any element of $K(X)$ has the form $[E_0] - [E_1]$ plus $[\infty^* E_0] = [\infty^* E_1]$ i.e. there is an isomorphism $\alpha\colon E_0\big|_{X\setminus L} \to E_1\big|_{X\setminus L}$ for some compact set L. Thus is it plausible that $K(X)$ consists of such length 2 complexes (E_0, E_1, α) as above.

101**. Definition: "External Product in K-Theory"**

Let X, Y be locally compact Hausdorff spaces and take $(E, \alpha) \in \mathscr{L}_1(X)$, $(F, \beta) \in \mathscr{L}_1(Y)$ now we have bundle over $X \times Y$ as:

$$(E \boxtimes F)_{(x,y)} := E_x \otimes F_y \tag{101.1}$$

Wlog take $n = 1$ then we have sequences

$$(E,\alpha)\colon 0 \longrightarrow E^0 \xrightarrow{\alpha^0} E^1 \longrightarrow 0 \tag{101.2}$$

$$(F,\beta)\colon 0 \longrightarrow E^0 \xrightarrow{\beta^0} E^1 \longrightarrow 0$$

with compact support. Now we could look at

$$0 \longrightarrow E^0 \boxtimes F^0 \xrightarrow{\alpha\boxtimes\beta} E^1 \boxtimes F^1 \longrightarrow 0 \tag{101.3}$$

$$0 \longrightarrow E^0_x \boxtimes F^0_y \xrightarrow{\alpha_x\boxtimes\beta_y} E^1_x \boxtimes F^1_y \longrightarrow 0$$

but this complex does not have compact support. Thus we must use a more complicated construction:

$$E \boxtimes F\colon 0 \longrightarrow E^0 \boxtimes F^0 \xrightarrow{\alpha\otimes I_{F^1}\oplus I_{E_0}\otimes\beta} E^1 \boxtimes F^0 \oplus E^0 \boxtimes F^1 \xrightarrow{I_{E^1}\otimes\beta\oplus\alpha\otimes I_{F^1}} E^1 \boxtimes F^1 \longrightarrow 0 \tag{101.4}$$

P Now $E \boxtimes F$ has compact support.

So it is not natural to always work with complexes of a fixed length. But of course we can again reduce $E \boxtimes F$ to length 1. Now there is a bilinear map:

$$K(X) \otimes K(Y) \xrightarrow{\boxtimes} K(X \times Y) \tag{101.5}$$

Assume next that $X, Y = V, W \in \operatorname{Vect}(M)$ then the diagonal map $\Delta\colon M \to M \times M$ induces by the pull-back a map $(\Delta^*)^*\colon K(V) \otimes K(W) \to K(V \oplus W)$. We may take $W = \{\, (x,0) \mid x \in M \,\} \simeq M$. Then we have the map $K(V) \otimes K(M) \to K(V)$, i.e. $K(V)$ is a $K(M)$ (bi-)module.

102. **Definition: "Homogeneous Complex"**

In order to define the notion of homogeneity we need to restrict ourselfs to a special kind of complexes. Let $V \in \operatorname{Vect}(M)$ with projection π and consider a complex (E,α) over V s.t. there are $\tilde{E}^i \in \operatorname{Vect}(M)$ with

$$E^i = \pi^* \tilde{E}^i \tag{102.1}$$

Hence $E^i_{\lambda v} = \{\lambda v\} \times E_{\pi(v)}$ and $\mathscr{L}(E^i_{\lambda v}, E^{i+1}_{\lambda v}) \simeq \mathscr{L}(\tilde{E}^i_{\pi(v)}, \tilde{E}^{i+1}_{\pi(v)})$ for any $v \in V$. We call (E,α) *homogeneous* of degree $m \in \mathbb{R}$ if

$$\alpha^i_{\lambda v} = \lambda^m \alpha^i_v \in \mathscr{L}(\tilde{E}^i_{\pi(v)}, \tilde{E}^{i+1}_{\pi(v)}) \tag{102.2}$$

We denote the set of all such complexes by ${}^m\mathscr{C}(V)$ and write ${}^m\mathscr{C}_\varnothing(V) := {}^m\mathscr{C}(V) \cap \mathscr{C}_\varnothing(V)$. As mentioned before in Remark 94 item 1 (Representatives of Bundle Complexes), it is sufficient to have such a complex defined outside a compact set, take e.g. the following

103**. Definition: "Sphere Bundle" and "Ball Bundle"**

Introduce now an Euclidean metric $\|\cdot\|$ on V and define for $R \in \mathbb{R}^+$:

$$B_R(V) := \{ v \in V \mid \|v\| \leq R \} \qquad \textit{ball bundle}$$
$$S_R(V) := \partial B_R(V) \qquad \textit{sphere bundle}$$

104**. Theorem:**

In addition to the isomorphisms in Theorem 99 we also have

$$ {}^m\mathscr{C}^n(V) \Big/ {}^m\mathscr{C}^n_{\emptyset}(V) \quad \simeq \quad \mathscr{C}^n(V) \Big/ \mathscr{C}^n_{\emptyset}(V) \quad \simeq \quad K(V) \tag{104.1}$$

for $m \geq 1$. So we can take representatives which are homogeneous of any such degree for K-theory.

Proof (104) Take a complex (E, α) of compact support, contained in $S_R(V)$ for some $R > 0$, and consider the pullback under the map

$$h \colon V \times [0,1] \ni (v,t) \mapsto \begin{cases} v & \|v\| \leq R \\ tR\frac{v}{\|v\|} + (1-t)v & \text{otherwise} \end{cases} \in V, \tag{104.2}$$

$$h^*(E, \alpha) \in \mathscr{C}^n(V \times [0,1]). \tag{104.3}$$

Then with $h_t(v) := h(t,v)$ we get $h_0 = \mathrm{id}_V$ and that h_1 is the projection onto $B_R(V)$. Thus we can conclude that $h_0^*(E,\alpha) = (E,\alpha)$ and that $h_1^*(E,\alpha)$ is homogeneous of degree 0. If we now use the scaling homotopy $(v,t) \mapsto t\|v\|^m v + (1-t)v$ we get a complex which is homogeneous of degree m and still homotopic to (E,α). So these complexes only depend on their restriction to $B_R(V)$ for R big enough.

105**. Theorem: Continuous and Smooth Vector Bundles**

Every topological vector bundle over a smooth manifold is equivalent to a smooth one. The in this way defined smooth structure is unique. Then for every continuous bundle map there is an equivalent smooth bundle map.

Proof (105) Let $E^{(N)} \to M$ be a topological vector bundle, i.e. there are $(g_{UV})_{U,V \in \mathcal{V}}$ with $\sigma_U \circ \sigma_V^{-1} = g_{UV} \in C(U \cap V, GL(N,\mathbb{C}))$ and wlog take $\mathcal{V}$ to be coordinate neighbourhood. Now choose smooth approximations $g_{UV}^\infty \in C^\infty(U \cap V, GL(N,\mathbb{C}))$ such that

$$g_{UV}^\infty = \sigma_U^\infty \circ (\sigma_V^\infty)^{-1} = \underbrace{\sigma_U^\infty \circ \sigma_U^{-1}}_{=: f_U} \circ \sigma_U \circ \sigma_V^{-1} \circ \underbrace{\sigma_V \circ (\sigma_V^\infty)^{-1}}_{=: f_V^{-1}} = f_U \circ g_{UV} \circ f_V^{-1} \tag{105.1}$$

This is an equivalent coclycle; hence defines a topological equivalent bundle. Bundle maps can be approximated similarily. Compare also [Hir76, Chapter 4 Theorem 3.5].

106**. Corollary: Smooth K-Theory**

In either construction of $K_{cpt}(M) = K(T^*M)$ in subsection II.2 (Some Elements of K-Theory) or subsection II.3 (K-Theory and Bundle Complexes) instead of starting with topological vector bundles $\mathrm{Vect}(T^*M)$ we may as well choose only smooth ones to get the same K-groups.

107. **Lemma: Logarithmic Law**

For $(E_0, E_1, \alpha), (E_1, E_2, \beta) \in K(X)$ we have

$$(E_0, E_1, \alpha) \oplus (E_1, E_2, \beta) = (E_0, E_2, \alpha\beta) \tag{107.1}$$

P **Proof** (107) Add the trivial bundle (E_1, E_1, id).

108. **Definition: "Analytic Index"**

Let M be a compact smooth manifold and $(E_0, E_1, \alpha) \in K_{cpt}(M)$, wlog α homogeneous of degree 0 and support contained in $M \subset T^*M$. Now choose any cut-off function ϕ with compact support and $\phi|_M = 1$. Then take the following *pseudodifferential operator* with *symbol* $\phi\alpha$

$$\mathrm{Op}(\phi\alpha)u(x) = (2\pi)^{-\frac{m}{2}} \int_{T_p^*M} e^{i\langle x,\xi\rangle} \phi\alpha(x, \xi)\hat{u}(\xi)\mathrm{d}\xi \tag{108.1}$$

Then we define the *analytic index* as

$$\mathrm{ind}_a \colon K_{cpt}(M) \to \mathbb{Z} \tag{108.2}$$

$$\mathrm{ind}_a(E_0, E_1, \alpha) := \mathrm{ind}\,\mathrm{Op}(\phi\alpha) \tag{108.3}$$

109. **Theorem:**

The *analytic index* is well-defined and a non-trivial ring homomorphism $K_{cpt}(M) \to \mathbb{Z}$.

Proof (109) We have to show that the index is independent of the choices made. Take any other choice of representing complexes (F^0, F^1, β). Then by the homotopy we can identify the bundles and get homotopic symbols α, β over the same bundles. Now their index is the same because of stability.

For complexes in $\mathscr{C}_\varnothing$, we get an invertible symbol, homogeneous of order 0 which defines an invertible operator in $L^2(M, E)$. Hence the index is 0.

From Lemma 107 (Logarithmic Law) we get additivity of the index:

$$\mathrm{ind}_a(\xi_1 \oplus \xi_2) = \mathrm{ind}_a \xi_1 + \mathrm{ind}_a \xi_2 \tag{109.1}$$

A. Misc

These parts need clean-up and have to be merged with the main part.

A.1. K-Theory

Assume now that $V \to X$ is a complex vectorbundle over a smooth manifold. Then there is a beautiful, important complex over V:

$$(\Lambda V)_v\colon 0 \longrightarrow \Lambda^0 V_{\pi(v)} \xrightarrow{i\,\mathfrak{w}(v)} \Lambda^1 V_{\pi(v)} \xrightarrow{i\,\mathfrak{w}(v)} \dots \xrightarrow{i\,\mathfrak{w}(v)} \Lambda^k V_{\pi(v)} \longrightarrow 0 \tag{A-1}$$

is exact for $v \neq 0$. It is called the *algebraic de Rham complex*, a complex of length k, represents the element λ_v in $K(V)$.

Take a elliptic operator $Q \in \Psi\mathrm{DO}_l(M, E^0, E^1)$ for a closed oriented (Riemannian) manifold with $T^*M \xrightarrow{\pi} M$. Then the symbol gives us a complex

$$0 \longrightarrow \pi^* E^0 \xrightarrow{\hat{Q}} \pi^* E^1 \longrightarrow 0$$

where $\hat{Q}$ is invertible on $T^*M \setminus M$ (i.e. outside a compact set).

Note: Above is a complex of length 1 Put $\pi^*E := \pi^*E^0 \oplus \pi^*E^1$ with grading $\varepsilon|_{\pi^*E^0} = \mathrm{id}$, $\varepsilon|_{\pi^*E^1} = -\mathrm{id}$. So we can regard $Q|_.\, \pi^*E \to \pi^*E \sim d := \begin{pmatrix} 0 & 0 \\ \hat{Q} & 0 \end{pmatrix}$, which satisfies $d^2 = 0$ and $\varepsilon d + d\varepsilon = 0$, i.e. d is *odd*. So we have a triple $\mathscr{E} := (E, \varepsilon, d)$ where (E, ε) is a superbundle and d is an odd differential. Now consider any such triple on T^*M and put

$$\mathscr{H}(\mathscr{E}) := {}^{\ker d}\!\big/_{\mathrm{im}\, d} = {}^{\ker d^+}\!\big/_{\mathrm{im}\, d^-} \oplus {}^{\ker d^-}\!\big/_{\mathrm{im}\, d^+} =: \mathscr{H}^+(\mathscr{E}) \oplus \mathscr{H}^-(\mathscr{E}) \tag{A-2}$$

Now assume that $\mathscr{H}(\mathscr{E}) = 0 \Rightarrow \ker d^+ = \mathrm{im}\, d^-$. So if d^+ (previously $\hat{Q}$) is invertible, then $d^- = 0$.

Remark If

$$\mathscr{E}_0\colon 0 \longrightarrow E^0 \xrightarrow{d^0} \dots \xrightarrow{d^{n-1}} E^n \longrightarrow 0$$

is any complex, then we can form again

$$E := \bigoplus_{i=0}^{n} E^i,\, d := \bigoplus_{i=0}^{n} d^i,\, \mathscr{E}|_{\bigoplus_{i\geq 0} E^{2i}} = \mathrm{id},\, \mathscr{E}|_{\bigoplus_{i\geq 0}} E^{2i+1} = \mathrm{id} \tag{A-3}$$

$$\Rightarrow \mathscr{E} = (E, d, \varepsilon) \text{ is a differential super bundle (dsb) and } \mathscr{H}(\mathscr{E}_0) \simeq \mathscr{H}^+(\mathscr{E}) \oplus \mathscr{H}^-(\mathscr{E}) \tag{A-4}$$

For X a locally compact Hausdorff space, $dsVect_{compt}(X) =$ isomorphism classes of $\{\mathscr{E} = (E, d, \varepsilon)\}$.

$$\phi\colon \mathscr{E} \to \mathscr{E}' \ \overset{isom}{homom} \Leftrightarrow \phi \in \overset{Iso}{Hom}\,(E, E'), d'\phi = \phi d, \varepsilon'\phi = \phi\varepsilon \tag{A-5}$$

Homotopy $\mathscr{E}_0 \sim \mathscr{E}_1$ iff there is (E, d, ε) over $I \times X$ s.t. $i_j^* E = E_j, j = 0, 1$.

Functoriality $X \xrightarrow{f} X' \leftarrow \mathscr{E}'$ then $f^*\mathscr{E}' \in dsVect_{comp}(X)$ only if f is *proper*. (i_{t_0} is proper for all t_0) E.g. $\pi: T^*M \to M$ is not proper since $\pi^{-1}(M) = T^*M$.

Operations

1. $\mathscr{E}_1 \oplus \mathscr{E}_2 := (E_1 \oplus E_2, \varepsilon_1 \oplus \varepsilon_2, d_1 \oplus d_2) =: (E, \varepsilon, d) \in dsVect_{comp}(X) with \operatorname{supp} \mathscr{E}_1 \oplus \mathscr{E}_2 \subset \operatorname{supp} \mathscr{E}_1 \cap \operatorname{supp} \mathscr{E}_2$ since $\mathscr{H}(\mathscr{E}_{1,x} \oplus \mathscr{E}_{2,x}) = \mathscr{H}(\mathscr{E}_{1,x}) \oplus \mathscr{H}(\mathscr{E}_{2,x})$.
2. $\mathscr{E}_1 \otimes \mathscr{E}_2 := (E_1 \otimes E_2, \varepsilon_1 \otimes \varepsilon_2, d_1 \otimes \mathrm{id}_{E_2} + \varepsilon_1 \otimes d_2) =: (E, \varepsilon, d)$. Then $d^2 = (d_1 \otimes \mathrm{id}_{E_2} + \varepsilon_1 \otimes d_2)(d_1 \otimes \mathrm{id}_{E_2} + \varepsilon_1 \otimes d_2) = d_1^2 \otimes \mathrm{id}_{E_2} + \mathrm{id}_{E_1} \otimes d_2^2 + d_1\varepsilon_1 \otimes d_2 + \varepsilon_1 d_1 \otimes d_2 = 0$.

110. **Lemma:**

1. If $\mathscr{E}_j \sim \mathscr{E}_j', j = 1, 2$ then $\mathscr{E}_1 \oplus \mathscr{E}_2 \sim \mathscr{E}_1' \oplus \mathscr{E}_2'$ and $\mathscr{E}_1 \otimes \mathscr{E}_2 \sim \mathscr{E}_1' \otimes \mathscr{E}_2'$.
2. $\mathscr{H}(\mathscr{E}_1 \otimes \mathscr{E}_2) \simeq \mathscr{H}(\mathscr{E}_1) \otimes \mathscr{H}(\mathscr{E}_2)$ (KÜNETH formula) So $\operatorname{supp} \mathscr{E}_1 \otimes \mathscr{E}_2 \subset \operatorname{supp} \mathscr{E}_1 \cup \operatorname{supp} \mathscr{E}_2$ and

$$E_1 \otimes E_2 = (E_1^0 \otimes E_2^0 \oplus E_1^1 \otimes E_2^1) \oplus (E_1^0 \otimes E_2^1 \oplus E_1^1 \otimes E_2^0) \tag{110.1}$$

Proof (110) Introduce metrics $h^{E_j}, j = 1, 2$ s.t. $e_j = e_j^*$ and now

$$d^* = d_1^* \otimes \mathrm{id}_{E_2} + \varepsilon_j \otimes d_2^* \tag{110.2}$$

Put

$$\begin{aligned} \Delta := d^*d + dd^* = \Delta_1 \otimes \mathrm{id}_{E_2} + \mathrm{id}_{E_1} \otimes \Delta_2 + d_1^*\varepsilon_1 \otimes d_2 + \varepsilon_j d_1 \otimes d_2^* \\ + d_1\varepsilon_j \otimes d_2^* + \varepsilon_1 d_1^* \otimes d_2 = \Delta_1 \otimes \mathrm{id}_{E_2} + \mathrm{id}_{E_1} \otimes \Delta_2 \end{aligned} \tag{110.3}$$

Recall the Hodge Theorem: $\mathscr{H}(\mathscr{E}) \simeq \ker \Delta$. So we get $\Delta = \langle \Delta_1 x_{j_0}, x_{j_0} \rangle + \lambda_{j_0} \|x_{j_0}\|^2 = \langle (\Delta_1 + \lambda_j) x_{j_0}, x_{j_0} \rangle \Gamma \lambda_{j_0} = 0$ and $\Delta_1 x_{j_0} = 0$.

$\mathscr{C}(X) =$ homotopy classes of isom. classes of dsb with compact support and $\mathscr{C}_\varnothing(X)$ are those with empty support.

111. **Theorem:**

$\mathscr{C}(X)$ is a commutative ring, $\mathscr{C}_\varnothing(X)$ is an ideal.

112. **Definition: “”**

$$K(X) := \mathscr{C}(X) \big/ \mathscr{C}_\varnothing(X) \tag{112.1}$$

is a commutative ring.

Remark If $\mathscr{E}_0$ and $\mathscr{E}_1$ are isomorphic, then they are also homotopic.

Proof (112) Represent them by pullbacks $\mathscr{E}_j = f_j^* \zeta, j = 0, 1$ and we have a homotopy $(f_t)_{t\in[0,1]}$ s.t. $f_j^* \zeta = \mathscr{E}_j$. This also works for dsb.

Remark If X is compact, we may consider all superbundles, in particular $0 \to E^0 \to 0 \to 0$. Thus, $[E^0]$ is well-defined and for any dsb $\mathscr{E}$ we get $[\mathscr{E}] = [E^0] - [E^1]$.

113**. Theorem:**

Let $\mathscr{E} = (E, \varepsilon, d)$ be as dsb with compact support. Then $[\mathscr{E}] \ni \mathscr{E}' = (E', \varepsilon', d')$ where

1. $\mathscr{E}'$ is smooth and there is $E_M \to M$ smooth, s.t. $E' = \pi^* E_M \Rightarrow E'_{(x,\lambda\xi)} \simeq E_{M,x}$ and $d'(x, \lambda\xi) = \lambda^m d'(x, \xi)$, for any $m \geq 0$.

114**. Lemma:**

If $\mathscr{E}_1, \mathscr{E}_2$ are dsb_{comp} over T^*M and $\operatorname{supp} \mathscr{E}_1 \cup \operatorname{supp} \mathscr{E}_2 \subset B_1(T^*M)$, then if $\mathscr{E}_1\big|_{B_1(T^*M)} = \mathscr{E}_2\big|_{B_1(T^*M)}$ we have $[\mathscr{E}_1] = [\mathscr{E}_2]$.

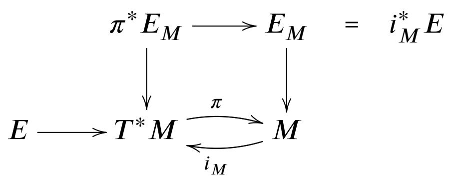
115**. Theorem:**

Every element a of $K(T^*M)$ for $\pi: T^*M \to M$ over a oriented Riemannian manifold, can represented by some $\mathscr{E} = (\pi^* E, \varepsilon^* \varepsilon_E, d)$ where $\varepsilon_M = (E, \varepsilon_E) \in \mathrm{Vect}_{s,\infty}(M)$ and d is a smooth differential such that

$$d(x, \xi) = |\xi|^l d(x, \frac{\xi}{|\xi|})(1 - \phi)(|\xi|) \tag{115.1}$$

where $l \in \mathbb{R}_+$ and $\phi \in C_c^\infty(\mathbb{R}_+)$ with $\phi = 1$ near 0.

Proof (115) By defining $\tilde{E} := i_0^* \tilde{E} = \pi^* E_M$ via the homotopy $i_t(v) := tv$ we have constructed the bundle.



We find $\mathscr{E}' = (E', \varepsilon', d') \in \mathrm{Vect}_{s,\infty}(T^*M)$ representing a. Then

- if $\operatorname{supp} \mathscr{E}' = \emptyset$ then we may assume that the homology is everywhere 0. Then we can deform $d_t'(x, \xi) = d'(x, t\xi)$, so a is represented by $(\pi^* \; E'\big|_M, \pi^* \; \varepsilon'\big|_M, \pi^*(d'\big|_M))$. This bundle is homogeneous of degree 0.

- Otherwise set

$$d_t(x,\xi) = |\xi|^l d(x, \frac{\xi}{|\xi|}) = d_t(x, |\xi|\frac{\xi}{|\xi|}) = |\xi|^{l(1-t)} d(x, |\xi|^t \frac{\xi}{|\xi|})(t\phi + (1-\phi))(|\xi|) \tag{115.2}$$

$$\Rightarrow d_0(x,\xi) = |\xi|^l d(x, \frac{\xi}{|\xi|})(1-\phi)(|\xi|) \tag{115.3}$$

where we assumed wlog that supp $\mathscr{E} \subset B_1(T^*M)$.

A.2. Topological Index

We look now at $K(TM)$ by taking a metric and using the induced isomorphism. Then $K(T_{pt}) = \mathbb{Z}$.

$$\begin{array}{c} K(T_{pt}) \simeq \mathbb{Z} \\ \Big\downarrow \phi_{\mathbb{C}^n}(\underline{n}) = \pi_{pt}^* \Lambda \mathbb{C}^N \otimes \pi_{pt}^* \underline{n} \\ K(\mathbb{C}^N) \\ \Big\downarrow \pi_{pt} \\ T_{pt} \end{array}$$

where ϕ is the Thom isomorphism.

$$\begin{array}{ccc} \pi_X^* \Lambda E & \longrightarrow & E \\ \Big\downarrow & & \Big\downarrow \pi_X \\ \Lambda E & \longrightarrow & X \end{array}$$

$$(\pi_X^* \Lambda E)_e = (\pi_X^* \Lambda^{ev} E)_e \oplus (\pi_X^* \Lambda^{odd} E)_e \tag{A-6}$$

Now the Clifford multiplication $c^{\Lambda E}(e)$ gives a map between these two parts.

P

Exercise Different metrics give homotopic complexes here.

We regard $\mathbb{C}^N$ as $T\mathbb{R}^N$. Then embedd (Whitney) M into $\mathbb{R}^N$ via $i_!$ and then $Ti_!: TM \to T\mathbb{R}^N$. (Due to Grothendieck) *i shriek*. But then $K(i_!): K(\mathbb{C}^N) \to K(TM)$ goes in the wrong direction because K is contravariant.

Simplest Case Set $M := \mathbb{R}^m$ which can be imbedded into any $\mathbb{R}^{m+k}$. Now $T\mathbb{R}^m = \mathbb{R}^m \oplus \mathbb{R}^m = \mathbb{C}^m$ and $T\mathbb{R}^{m+k} = \mathbb{R}^{m+k} \oplus \mathbb{R}^{m+k} = \mathbb{R}^m \oplus \mathbb{R}^m \oplus \mathbb{R}^k \oplus \mathbb{R}^k = T\mathbb{R}^m \oplus \mathbb{C}^k =: E^k \to T\mathbb{R}^m$. So the tangent bundle of the manifold we embed in, is a complex vector bundle over the original tangent bundle. Then

$$K(T\mathbb{R}^{m+k} = K(E^k) \xleftarrow{\phi_{E^k}} K(T\mathbb{R}^m) \tag{A-7}$$

Then we can define an index as

$$\mathrm{ind}_t := \phi^{-1}_{\mathbb{C}^{m+k}} \phi_{\mathbb{C}^k} \tag{A-8}$$

116**. Lemma: Tangent bundles of vector bundles**
Let $E \to M$ be any vector bundle, then

$$TE \simeq TM \oplus E \oplus E \tag{116.1}$$

Proof (116) Look at $TM \oplus E \oplus E$ as a trivial bundle over E so we could define a isomorphism
P $TE \simeq (\pi^E)^*(TM \oplus E)$.
Exercise!

Consider now the general case Take again $M \hookrightarrow \mathbb{R}^{m+k}$. We can split $T\mathbb{R}^{m+k}\big|_M = TM^m \oplus N$ where N is the normal bundle, defined by a certain metric.

As M is closed, there is an open tubular neighbourhood $T_\varepsilon M \subset \mathbb{R}^{m+k}$ around M. But $T_\varepsilon M \simeq B_\varepsilon(N) \simeq N$. Thus TN can be viewed as an open subset of $T\mathbb{R}^{m+k}$.

117**. Lemma:**
$K(X) = \ker(i^*_{pt}\colon K(\dot{X}) \to \mathbb{Z})$

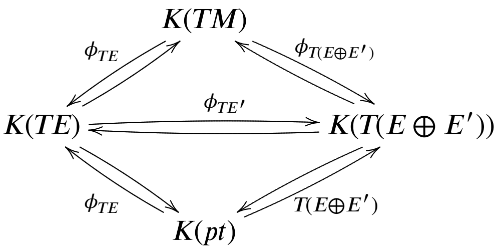
Proof (117) See [Seg68].

So together with the restriction map $i_*\colon K(U) \to K(X)$ we get $K(TM) \xrightarrow{\phi_E} K(E) \xrightarrow{i_*} K(T\mathbb{R}^{m+k}) \xrightarrow{\phi^{-1}(\mathbb{R}^m+k)} K(\mathbb{C}^{m+k}) \xrightarrow{\phi^{-1}_{\mathbb{C}N}} K(T_{pt})$.

118**. Lemma:**
Let $M^m \hookrightarrow N^n$ be a smooth embedding. This defines $i_!\colon K(TM) \to K(TN)$ such that $(ji)_! = j_! i_!$.

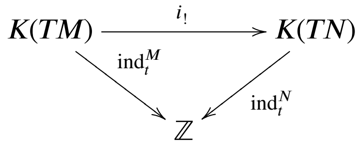
119**. Lemma:**
Given two embeddings $i^{(\prime)}\colon M \to E^{(\prime)} := \mathbb{R}^{m+k^{(\prime)}}$ then the topological index functions coincide.

Proof (119) Consider $i \oplus i'\colon M \to E \oplus E'$ and the family $i \oplus si'$, $s \in [0,1]$. This gives the following diagramm:



120**. Lemma: Properties**

1. $\operatorname{ind}_t\big|_{K(T_{pt})} = \operatorname{id}_{\mathbb{Z}}$
2. ind_t commutes with $i_!$.



Now every homomorphism with these properties actually coincides with the topological index. Proving the second property for the analytical index is really hard.

A.3. Another Look at ψdos

Local theory in open $U \subset \mathbb{R}^m$.

$$\operatorname{Sym}^m(U) = \left\{ p(x,\xi)\colon U \times \mathbb{R}^m \to \mathbb{C} \,\Big|\, \sup_{\substack{x\in K \\ K\subset U \text{ compact}}} |D_\xi^\alpha D_x^\beta p(x,\xi)| \le C_{K,\alpha,\beta} \langle \xi \rangle^{m-|\alpha|} \right\} \tag{A-9}$$

Define an operator

$$p(x,D)u(x) := \operatorname{Op}(p)u(x) := \int_{\mathbb{R}^m} e^{i\langle x,\xi\rangle} p(x,\xi)\hat{u}(\xi)\, đ\xi, \quad đ\xi := (2\pi)^{-\frac{m}{2}} d\xi \tag{A-10}$$

which is linear $\mathscr{D}(U) \to \mathscr{E}(U)$

121**. Definition: "Pseudodifferential Operator"**

We define a space of operators $\Psi\mathrm{DO}^m(U)$ as follows

1. $P\colon \mathscr{D}(U) \to \mathscr{E}(U)$ is continuous
2. for $f \in \mathscr{D}(U)$ the *commutator*

   $$p_f(x,\xi) := e^{-i\langle x,\xi\rangle} P(f e^{i\langle\cdot,\xi\rangle}) \in \mathcal{S}^m(U) \tag{121.1}$$

   for all $(x,\xi) \in U \times \mathbb{R}^m$.
3. We define $\Psi\mathrm{CDO}^m(U)$ as the subset of $P \in \Psi\mathrm{DO}^m(U)$ with the property that $p_f \in \operatorname{Sym}^m_{\mathrm{cl}}(U)$, $f \in \mathscr{D}(U)$ and

   $$q \in \operatorname{Sym}^m_{\mathrm{cl}}(U) \Leftrightarrow \lim_{t\to\infty} t^{-m} q(x,t\xi) =: \hat{q}(x,\xi) \neq 0 \tag{121.2}$$

P

   We then define $\hat{p}(x,\xi) := \hat{p}_f(x,\xi)$ for $f \in \mathscr{D}(U)$, $f(x) = 1$. This is well-defined.

122. **Theorem:** SEELEY, HÖRMANDER, KOHN-NIRENBERG
The spaces $\Psi\mathrm{DO}^m(U)$ and $\Psi\mathrm{CDO}^m(U)$ are invariant under diffeomorphisms. In particular

$$\hat{p}(x,\xi) = \hat{p}(\psi(x), d\psi(x)(\xi)) \tag{122.1}$$

for $p \in \Psi\mathrm{CDO}(U)$ and $\psi \in \mathrm{Diff}(U)$. So $\hat{p}$ is invariantly defined on T^*U.

Now we have fairly obvious extensions to compact manifolds M^m and vector bundles $E, F \to M$, to get.

$$\Psi\mathrm{CDO}^m(M,E,F) \tag{A-11}$$

In particular $\hat{p}$ is now invariantly defined on $T^*M \setminus M$ and $\hat{p}(x,\xi) \in \mathscr{L}(\pi^* E_{(x,\xi)}, \pi^* F_{(x,\xi)})$ and invertible iff P is elliptic.

123. **Theorem:**
Let $P, Q \in \Psi\mathrm{CDO}(M,E,F)$ then

1. $\widehat{PQ} = \hat{P}\hat{Q}$
2. if there are Hermitian metrics h^E, h^F and a Riemannian metric g^{TM} then $\widehat{P^*} = \hat{P}^*$.
3. For any $m \in N$ there is an exact sequence:

$$0 \longrightarrow \Psi\mathrm{CDO}^{m-1}(M,E,F) \hookrightarrow \Psi\mathrm{CDO}^m(M,E,F) \xrightarrow{\hat{}} \mathrm{Sym}^m(M,E,F) \longrightarrow 0 \tag{123.1}$$

$P \in \Psi\mathrm{CDO}(M,E)$ is bounded in H^s iff $l \le s$, compact if $l < s$ and trace class if $l < s - m$.

A.4. The Topological Index

We need to find another natural homomorphism $K(T^*M) \to \mathbb{Z}$ which equals ind_a.

124. **Lemma:**
$K(X) = \tilde{K}(\dot{X}) = \ker(i^*_{pt}\colon K(X) \to \mathbb{Z} = K(pt)$

Drawback For a non-compact manifold N, $\dot{N}$ is not a manifold in general. Note that

$$(T^*M)^{\cdot} = \overline{B_1(T^*M)} \big/ S_1(T^*M) \tag{A-12}$$

Note T^*M can be replaced by any real (smooth) vector bundle over M. Let V, W be vector bundles over M. Then we can form the exterior tensor product $V \boxplus W \to M \times N$ with fibre $(V \boxplus W)_{(x,y)} = V_x \otimes W_y$.

Seminar Tasks

- Proof BOTT's periodicity theorem for complex K-theory and show the Thom isomorphism.
- Descriptions of $\mathbb{C}$-vector bundles and notion of equivalence
  - as fibre bundles, using cocycles and their equivalence [STEENROD]
  - as continuous maps $f\colon X \to \mathrm{Gr}^n(\mathbb{C}^\infty)$ into the Grassmannian, using homotopy equivalence [MILNOR?]
  - as idempotents in finitely generated projective modules over $C^{(\infty)}(X)$, using conjugation [SERRE-SWAN-Theorem, CONNES]

Problems

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