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Functional Analysis

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XeTeX The code was written in gvim with the help of vim-latex and compiled with xelatex, all on Gentoo Linux. Thanks to the free software community and to the fellow TeX users on TeX.SX for their great help and advice.
If you want to make any comments or suggest corrections to this script, please contact me! You can either email me directly or use the contact form on my website.

Contents

1 Seminorms on Vector Spaces over $\mathbb{K}$ (where $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$)

In the following, let V be a $\mathbb{K}$-vector space.

1 **Definition: "Seminorm, Norm"**

A map $\|\cdot\|: V \to \mathbb{R}_+ := [0, \infty), x \mapsto \|x\|$ which satisfies

1. triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$
2. positive homogeneity: $\forall \alpha \in \mathbb{K}, x \in V: \|\alpha x\| = |\alpha|\|x\|$

is called a *seminorm* on V and $(V, \|\cdot\|)$ is called a *seminormed vector space*. If moreover

3. $\forall x \in V: \|x\| = 0 \Leftrightarrow x = 0$

then $\|\cdot\|$ is called a *norm* and $(V, \|\cdot\|)$ is a *normed vector space*.

Examples of Normed Vector Spaces

1. For any p ($1 \leq p < \infty$) the vector space $\mathbb{K}^n$ has the norms

$$\|(\alpha_1, \alpha_2, \ldots, \alpha_n)\|_p := \sqrt[p]{|\alpha_1|^p + |\alpha_2|^p + \ldots + |\alpha_n|^p}$$

$$\|(\alpha_1, \alpha_2, \ldots, \alpha_n)\|_\infty := \max_{1 \leq j \leq n} |\alpha_j| \overset{!}{=} \lim_{p \to \infty} \|(\alpha_1, \alpha_2, \ldots, \alpha_n)\|_p$$

2. For a compact subset $X \subset \mathbb{K}^n$ the set of all continuous functions over X, $C(X, \mathbb{K})$ is normed by

$$\|f\|_\infty := \sup_{x \in X} |f(x)|$$

3. We get a seminorm on $\mathcal{L}_p(\mathbb{R}^n)$ as

$$\|f\|_p := \sqrt[p]{\int_{\mathbb{R}^n} |f(x)|^p \mathrm{d}x}$$

A norm is introduced when we pass on to the space of cosets $L_p(\mathbb{R}^n)$. This principle works in any case of a seminorm for V/V_0 with $V_0 := \{ x \in V \mid \|x\| = 0 \}$ and

$$\|\bar{x}\|_{V_0} := \inf_{y \in V_0} \|x + y\|$$

2 **Definition: "Semimetric, Metric"**

Let X be a set and $\rho: X \times X \to \mathbb{R}_+$ a map. ρ is a *semimetric* on X if it satisfies:

1. $\forall x \in X: \rho(x, x) = 0$
2. $\forall x, y, z \in X: \rho(x, z) \leq \rho(x, y) + \rho(y, z)$
3. $\forall x, y: \rho(x, y) = \rho(y, x)$

We call (X, ρ) a *semimetric space*. If additionally

4. $\forall x, y \in X: \rho(x, y) = 0 \Leftrightarrow x = y$

holds, ρ is a *metric* and (X, ρ) is a *metric space*.

3 **Lemma: (semi)metrics induced by (semi)norms**

If $\|\cdot\|$ is a (semi)norm on V, then $\rho_{\|\cdot\|}(x,y) := \|x-y\|$ is a (semi)metric.

4 **Definition: "Seperability"**

A metric space (X,ρ) is called *separable* if there is a countable, dense subset.

5 **Definition: "Isometry, Lipschitz"**

Let $(X,\rho), (Y,\rho')$ be metric spaces. A map $f\colon X \to Y$ is called *isometric* if

$$\forall x,y \in X\colon \rho'(f(x),f(y)) = \rho(x,y)$$

f is called *Lipschitz*, if

$$\exists C < \infty \forall x,y \in X\colon \rho'(f(x),f(y)) \leq C\rho(x,y)$$

6 **Remark: Range of Normed Spaces**

Every separable normed space $(V,\|\cdot\|)$ admits a linear isometric embedding $i\colon (V,\|\cdot\|) \to (C([0,1]),\|\cdot\|_\infty)$.

7 **Definition: "Banach Space, Hilbert Space"**

A *Banach space* is a normed vector space $(V,\|\cdot\|)$ that is complete with respect to its metric $\rho_{\|\cdot\|}$. Does it satisfy the parallelogram equation:

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

it is called a *Hilbert space*.

8 **Remark: Completeness of Metric Spaces**

A metric space (X,ρ) is a topological space at the same time. On the other hand we can also define Cauchy sequences as sequences $(x_i)_{i\in\mathbb{N}} \in X^{\mathbb{N}}$ with

$$\forall \varepsilon > 0 \exists k \in \mathbb{N} \forall n,m > k\colon \rho(x_n,x_m) < \varepsilon$$

(X,ρ) is called *complete* if every Cauchy sequence is convergent with its limit contained in X.

9 **Definition: "Boundedness, Functional"**

Let $T\colon V \to W$ be a linear map between two seminormed spaces $(V,\|\cdot\|_V)$ and $(W,\|\cdot\|_W)$. T is *bounded* if there is a value $C < \infty$ so that

$$\forall x \in V\colon \|T(x)\| \leq C\|x\|$$

In the case $W = \mathbb{K}$ such a bounded map T is called a (linear) *functional* on V.

10 **Lemma: Inverse Triangle Inequality**

For any $x,x',y,y' \in X$ in a metric space (X,ρ)

$$|\rho(x,x') - \rho(y,y')| \leq \rho(x,y) + \rho(x',y')$$

holds.

Open Balls We define a base of a topology over X via the ε-balls

$$U(x,\varepsilon) := \{\ y \in X \mid \rho(x,y) < \varepsilon\ \}$$

11 **Lemma: Properties of the Topological Base**

For these base sets the following conditions hold

1. $\forall x \in X, \varepsilon > 0 : x \in U(x,\varepsilon)$
2. $\forall x \in X, \varepsilon_1, \varepsilon_2 > 0 : \varepsilon_1 \leq \varepsilon_2 \Rightarrow U(x,\varepsilon_1) \subset U(x,\varepsilon_2)$
3. $\forall x \in X, y \in U(x,\varepsilon), \varepsilon > 0 : U(y, \varepsilon - \rho(x,y)) \subset U(x,\varepsilon)$

12 **Definition: "Open Sets"**

A subset $W \subset X$ is called *open* if W is the union of suitable ε-balls. In a metric space this is equivalent to the condition, that for each $x \in W$ there are $\varepsilon > 0$ and $U(x,\varepsilon) \subset W$.

13 **Definition: "Topology"**

A general *topology* on a set X is a system $\mathcal{O}$ of subsets of X which satisfies

1. $\emptyset, X \in \mathcal{O}$
2. $\forall A, B \in \mathcal{O} : A \cap B \in \mathcal{O}$
3. For any family of subsets $(W_i)_{i \in I}$ of $\mathcal{O}$ we have $\bigcup_{i \in I} W_i \in \mathcal{O}$.

14 **Definition: "Hausdorff"**

A topological space $\mathcal{O}$ is called *Hausdorff* if every two points have disjoint neighbourhoods. i.e.

$$\forall x, y \in X, x \neq y \exists U_x, U_y \in \mathcal{O} : x \in U_x \wedge y \in U_y \wedge U_x \cap U_y = \emptyset$$

15 **Lemma: Induced Topology**

Let (X,ρ) be a metric space, then the familiy of open subsets of X defines a Hausdorff topology.

16 **Definition: "Continuous Map"**

A map between two topological space $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, $f : X \to Y$ is called *continuous* if the inverse image of any open set in Y is open in X. A map is called continuous according to a metric if it is continuous according to the induced topology.

As any seminorm induces a semimetric, which in turn induces a topology, we get a canonical notion of continuous maps on seminormed vector spaces and semic space.

17 **Definition: "Locally Continuous Map"**

f is *continuous* in $x \in X$ according to a metric ρ_Y if

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x' \in U(x,\delta) : \rho_Y(f(x), f(x')) < \varepsilon$$

18 **Definition: "Uniformly Continuous Map"**
$f\colon X \to Y$ is *uniformly continuous* if

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x \in X, x' \in U(x,\delta)\colon \rho_Y(f(x), f(x')) < \varepsilon$$

19 **Definition: "Hölder Continuous Map"**
$f\colon X \to Y$ is *Hölder continuous* if there is a $C < \infty$ and $\alpha > 0$ such that

$$\forall x, x' \in X\colon \rho_Y(f(x), f(x')) \leq C \rho_X(x, x')^\alpha$$

Example We define a metric on $X \times Y$ with some metrics ρ_X and ρ_Y:

$$\rho_{X\times Y}((x,y),(x',y')) := \rho_X(x,x') + \rho_Y(y,y')$$

According to Lemma 10 (Inverse Triangle Inequality) the function $\rho\colon X\times X \to [0,\infty)$ is Lipschitz continuous with Lipschitz constant $C \leq 1$.

20 **Lemma: Continuity on Metric Spaces**
Let $f\colon (X,\rho_X) \to (Y,\rho_Y)$ be metric spaces and $x_0 \in X$. Then the following statements are equivalent:

1. f is continuous at x_0.
2. For any sequence $(x_n)_{n\in\mathbb{N}}$: $\lim_{n\to\infty} x_n = x_0 \Rightarrow \lim_{n\to\infty} f(x_n) = f(x_0)$.

21 **Lemma: Local and Global Continuity**
A map f between metric spaces X and Y is (globally) continuous on X iff it is (locally) continuous at every $x \in X$.

22 **Lemma: Continuity of Linear Maps between Vector Spaces**
Let $(V, \|\cdot\|_V), (W, \|\cdot\|_W)$ be seminormed vector spaces and $T\colon V \to W$ a linear map. Then the following conditions are equivalent:

1. T is continuous at some $v_0 \in V$
2. For some $v_0 \in V$ and any sequence $(v_n)_{n\in\mathbb{N}}$: $\lim_{n\to\infty} v_n = v_0 \Rightarrow \lim_{n\to\infty} f(v_n) = f(v_0)$.
3. T is continuous at $0 \in V$.
4. T is continuous on V.
5. T is uniformly continuous on V.
6. T is Lipschitz with respect to the metrics $\rho_{\|\cdot\|_V}$ and $\rho_{\|\cdot\|_W}$.
7. T is bounded.

23 **Definition: "Operator Norm"**

Let $T\colon (V, \|\cdot\|_V) \to (W, \|\cdot\|_W)$ be a bounded linear map between two normed vector spaces, we define the *operator norm* as

$$\|T\|_{\mathcal{B}} := \sup_{\|x\|_V \le 1} \|Tx\|_W$$

By $\mathcal{B}$ or $\mathcal{B}(V, W)$ we denote the space of all bounded linear maps from V to W.

24 **Lemma: Vectorspace $\mathcal{B}$**

For $\mathbb{K}$-vector spaces V and W the space $\mathcal{B}(V, W)$ is a normed $\mathbb{K}$-vector space with the canonical addition, scalar multiplikation and the operator norm. Let $S, T \in \mathcal{B}(V, W)$

$$\forall v \in V\colon (S + T)(v) := S(v) + T(v)$$
$$\forall v \in V, \alpha \in \mathbb{K}\colon (\alpha T)(v) := \alpha T(v)$$

25 **Remark: Operator Norm of Unbounded Linear Maps**

The set of all the linear maps $T\colon V \to W$ is denoted by $\mathrm{Lin}(V, W)$. One can easily see that $\forall T \in \mathrm{Lin}(V, W)\colon T \in \mathcal{B}(V, W) \Leftrightarrow \|T\|_{\mathcal{B}} < \infty$. If we define $0 \cdot \infty := 0 \wedge (t > 0 \Rightarrow t \cdot \infty = \infty)$, $\mathrm{Lin}(V, W)$ forms a normed vector space too.

2 Criteria for Closedness and Completeness

26 **Lemma: Closure**

In any topological space X, the closure $\mathrm{cl}(A)$, denoted mostly as $\bar{A}$, of a set A is the union of A with its boundary points. Which takes the form

$$\bar{A} = \left\{ x \in X \;\middle|\; \exists (x_n)_{n\in\mathbb{N}}, x_n \in A\colon \lim_{n\to\infty} x_n = x \right\}$$

in metric spaces.

27 **Definition: "Equivalence of Norms"**

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space V are called *equivalent* if there are some $\gamma > 0, C < \infty$ so that

$$\forall v \in V\colon \gamma \|v\|_1 \le \|v\|_2 \le C\|v\|_1$$

28 **Lemma: Norms on $\mathbb{K}^n$**

All the norms on $\mathbb{K}^n$ are equivalent.

Proof (28) Given one particular norm $\|\cdot\|$ we have

$$\forall v \in \mathbb{K}^n\colon \|v\| \le C\|v\|_2 \quad \text{with} \quad C := \sqrt{\sum_{j=1}^{n} \|e_j\|^2}$$

and on the other hand the boundary of the $\|\cdot\|_2$ unit ball is compact in $\mathbb{K}^n$ which gives us $\inf\left\{ \|v\| \;\middle|\; \|v\|_2 = 1 \right\} > 0$ because $\|\cdot\|$ is continuous. So we get $\gamma\|\cdot\|_2 \le \|\cdot\|$ for some fixed γ.

29 **Lemma: Condition for Norm Equivalence**

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are equivalent iff $\mathcal{O}_{\|\cdot\|_1} = \mathcal{O}_{\|\cdot\|_2}$ (i.e. they induce the same topology).

30 **Lemma: Completeness of Subsets**

Any subset of a complete metric space is closed iff it is complete.

31 **Corollary: Completeness of Isometry Image**

If $T\colon B \to V$ is a linear, isometric map from a Banach space into a normed space ($\forall b \in B\colon \|T(b)\|_V = \|b\|_B$), then $T(B)$ is a closed linear subspace of V.

32 **Theorem: Completion Theorem**

Let (X, ρ) be a metric space. Then there is a metric space $(\hat{X}, \hat{\rho})$ and an isometric embedding $i\colon X \hookrightarrow \hat{X}$ such that $i(X)$ is dense in $\hat{X}$ and $\hat{X}$ is complete. $(\hat{X}, \hat{\rho})$ is called a completion of (X, ρ). The embedding is unique up to isometry.

33 **Lemma: Completion of Normed Space**

Let $(V, \|\cdot\|)$ be a normed vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, then the completion $(\hat{V}, \hat{\rho_{\|\cdot\|}})$ of $(V, \rho_{\|\cdot\|})$ has a unique $\mathbb{K}$-vector space structure such that $i\colon V \to \hat{V}$ is linear and $\|v\|_{\hat{V}} := \hat{\rho_{\|\cdot\|}}(0, v)$ defines a norm on $\hat{V}$. $(\hat{V}, \|\cdot\|_{\hat{V}})$ is a Banach space and $i\colon V \to \hat{V}$ is a linear isometric map from V into $\hat{V}$.

Proof (33) Let $\alpha \in \mathbb{K}$, $y \in \hat{V}$. Find $(x_n) \subset V$ with $\lim_{n\to\infty} i(x_n) = y$. That gives us $(x_n) \subset V$ is Cauchy and therefore $(\alpha x_n) \subset V$ is Cauchy too so that $\lim_{n\to\infty} i(\alpha x_n)$ exists in $\hat{V}$ and we define $\alpha y := \lim_{n\to\infty} i(\alpha x_n)$. We have to check that the definition of αy is independent of (x_n) and that $\hat{\rho}(0, \alpha y) = |\alpha|\hat{\rho}(0, y)$.

Then we define $y + y' := \lim_{n\to\infty} i(x_n + x_n')$ likewise. We have to check the algebraic rules and $\hat{\rho}(0, x + y) \le \hat{\rho}(0, x) + \hat{\rho}(0, y)$. We also get $\hat{\rho}(x + z, y + z) = \hat{\rho}(x, y)$.

3 Compact and Precompact Metric Spaces

34 **Definition: "Compactness"**

Let $(X, \mathcal{O})$ be a topological space. $Z \subset X$ is called *compact* if every covering of Z by a family of open sets contains a finit subcovering. If the closure of Z is compact in X, Z is called *subcompact*.

If we have a metric ρ on X, we can define: $(Z, \rho|_{Z\times Z})$ is *sequentially compact* if every sequence $(x_n) \subset X$ contains a subsequence that converges in Z. Z is called *precompact* if for any $\varepsilon > 0$ there are finit many ε-neighbourhoods of points in Z which cover Z.

35 **Lemma: Compactness Lemma**

If (X, ρ) is a metric space the following conditions hold:

1. X is compact iff it is sequentially compact.
2. If $(Z, \rho|_{Z\times Z})$ is subcompact and complete, then Z is compact in X.
3. If $(Z, \rho|_{Z\times Z})$ is precompact and (X, ρ) is complete, then Z is subcompact in X.
4. (X, ρ) is compact iff (X, ρ) is precompact and complete.
5. Each compact subset $Z \subset X$ is closed in X.
6. Every closed subset of a compact space (X, ρ) is compact.
7. $Z \subset X$ is precompact iff $\bar{Z}$ is precompact.

36 **Remark: Precompact Subsets**

Any subset of a precompact set is precompact.

3.1 Precompactness of Function Sets

37 **Definition: "Diameter, Bounding"**

Let (X, ρ) be a metric space. Then we define $\delta\colon \wp(X) \to [0, \infty]$ via

$$\delta(Z) := \sup_{z,z' \in Z} \rho(z, z')$$

and call it the *diameter* of Z. Z is called *bounded* if $\delta(Z) < \infty$.

38 **Lemma: Image of Compacta**

Let $(E, \mathcal{O})$ be a compact topological space and $f \in C(E, X)$ with a metric space (X, ρ) then

1. $f(E)$ is compact.
2. $f(E)$ is complete, closed and bounded.

Proof (38) Compactness is trivial, completeness and closedness follow from it according to Lemma 35 (Compactness Lemma). Boundedness is easily shown because for a fixed point $e_0 \in E$ the function $e \mapsto \rho(f(e_0), f(e))$ assumes its maximum.

39 **Remark: Compactness Conditions**

For a metric space (X, ρ) the following equivalences hold:

1. X is precompact iff every sequence $(x_n)_n \in X^{\mathbb{N}}$ contains a Cauchy subsequence $(x_{n_k})_k$.

2. X is precompact iff

$$\forall \varepsilon > 0 \exists \text{finit } Z \subset X \forall x \in X \colon \rho'(x, Z) := \inf_{z \in Z} \rho(x, z) < \varepsilon$$

Z is called an ε-net for (X, ρ).

3. X is compact iff every sequence contains a convergent subsequence.

4. For every subset S of a complete metric space X:
   a) S is compact iff S is precompact and closed.
   b) S is precompact iff S is subcompact.

40 Remark: Precompactness under Lipschitz Operators

If $\varphi \colon (X, \rho) \to (Y, \rho')$ is Lipschitz then φ maps precompact subsets of X to precompact subsets of Y.

41 Definition: "Supremum Metric, Equicontinuity"

Let E be a topological space, (X, ρ) a metric space and $C(E, X)$ the space of all continuous maps from E to X. If E is compact we introduce the *supremum metric* on $C(E, X)$ as

$$\rho_\infty \colon C(E, X) \times C(E, X) \to [0, \infty), (f, g) \mapsto \sup_{e \in E} \rho(f(e), g(e))$$

A subset $H \subset C(E, x)$ is called *equicontinuous* if

$$\forall e \in E, \varepsilon > 0 \exists U_e \in \mathcal{O}(E), e \in U_e \forall f \in H, c \in U_e \colon \rho(f(c), f(e)) < \varepsilon$$

42 Theorem: Arzela-Ascoli

Let E denote a compact topological space and (X, ρ) a complete metric space. Then $H \subset C(E, X)$ is precompact iff

1. H is equicontinuous

2. Any $H(e) := \{ f(e) \mid f \in H \}, e \in E$ has a compact closure.

43 Corollary: Classical Arzela-Ascoli

Let $X = \mathbb{R}$, then H is precompact iff

1. H is equicontinuous

2. $H(x)$ is bounded for any $x \in X$.

Proof (42) Assume that $H \subset C(E, X)$ is precompact and let $e \in E, \varepsilon > 0$ be given. Then we can find an $\frac{\varepsilon}{3}$-net $Z = \{f_1, \dots, f_n\} \subset H$ with $\forall f \in H\colon \min_r \rho_\infty(f, f_r) < \frac{\varepsilon}{3}$. Let

$$U_e := \left\{ c \in E \,\middle|\, \forall i = 1, \dots, n\colon \rho(f_i(c), f_i(e)) < \frac{\varepsilon}{3} \right\}$$

U_e is open and contains e so that $\forall f \in H\colon \rho(f(c), f(e)) < \varepsilon$. That means H is equicontinuous.

Now $H(e) := \{ f(e) \mid f \in H \}$ is the image of H under the map $\chi_e\colon f \in C(E, X) \mapsto f(e) \in X$. We have $Lip(\chi_e) = 1$. That means $\chi_e(Z) \subset H(e)$ is an ε-net for $H(e)$.

For the other direction we fix an $\varepsilon > 0$ and choose $U_e \in \mathcal{O}(E)$ in such a way that H is equicontinuous:

$$\forall c \in U_e, f \in H\colon \rho(f(c), f(e)) < \frac{\varepsilon}{3}$$

Since E is compact there are finitely many $e_1, \dots, e_n \in E$ for which $E = U_{e_1} \cup \dots \cup U_{e_n}$. Now every $H(e_j)$ is subcompact in the complete space (X, ρ) and so it is precompact:

$$\exists x_1^{(j)}, \dots, x_{n_j}^{(j)} \in X \forall f \in H\colon \inf_i \rho(f(e_i), f(x_k^{(i)})) < \frac{\varepsilon}{6}$$

Thus for each $f \in H$ there exists a map $\gamma\colon \{1, \dots, n\} \to F_1 \cup \dots \cup F_n$ with $F_j := \{x_1^{(j)}, \dots, x_{n_j}^{(j)}\}$ and $\gamma(j) \in F_j$ and $\rho(f(e_j), \gamma(j)) < \frac{\varepsilon}{6}$.

Then we examine $M_\gamma := \left\{ f \in H \,\middle|\, \rho(f(e_j), \gamma(j)) < \frac{\varepsilon}{6} \right\}$ and see $H = \bigcup_\gamma M_\gamma$.

Let $f, g \in M_\gamma$, $c \in E \Rightarrow c \in U_{e_j}$ for some $j = \{1, \dots, n\}$, so we get

$$\rho(f(c), g(c)) \le \rho(f(c), f(e_j)) + \rho(f(e_j), g(e_j)) + \rho(g(c), g(e_j)) < \frac{2\varepsilon}{3} + \rho(f(e_j), g(e_j))$$
$$\rho(f(e_j), g(e_j)) \le \rho(f(e_j), \gamma(j)) + \rho(g(e_j), \gamma(j)) < \frac{\varepsilon}{3}$$
$$\Rightarrow \forall f, g \in M_\gamma\colon \rho(f(c), g(c)) < \varepsilon \Rightarrow \rho_\infty(f, g) \le \varepsilon$$

Choose one $f_\gamma \in M_\gamma \subset H$. Then we have $M_\gamma \subset U(f_\gamma, 2\varepsilon)$ which implies $H \subset \bigcup_\gamma U(f_\gamma), 2\varepsilon$ so that $\{f_\gamma\}$ is an 2ε-net and H is shown as precompact.

The advantage of this way to prove the theorem is, that we did not need separability.

4 Baire Category Theorem

44 **Definition: "Dense, Meager"**

Let X be a topological space. Then $A \subset U$ is *dense* iff $\bar{A} = X$, *nowhere dense* in X iff $\operatorname{Int} \bar{A} = \emptyset$ and*meager* iff it is a countable union of nowhere dense subsets of X.

45 **Lemma: Conditions for Nowhere Density**

Let (X, ρ) be a metric space

1. $A \subset X$ is nowhere dense iff $\bar{A}$ does not contain any open ball $U(x, \varepsilon), x \in X, \varepsilon > 0$.

2. If $A \subset X$ is closed it is nowhere dense iff

$$\forall x \in X, \varepsilon > 0\colon (X \setminus A) \cap U(x, \varepsilon) \neq \emptyset$$

3. If (X, ρ) is complete any descending chain of ε_n-Balls with $\lim_{n\to\infty} \varepsilon_n = 0$ contains exactly one unique $x \in X$:

$$E_1 \supset E_2 \supset \ldots \Rightarrow \bigcap_{j=1}^{\infty} E_j = \{x\}$$

46 **Theorem: Special Baire Category Theorem**

Let (X, d) be a complete metric space and $A_i \subset X, i = 1, 2, \ldots$ closed subsets of X so that their union has a not empty interior, then some A_i has an not empty interior:

$$\operatorname{Int}(\bigcup_{n=1}^{\infty} A_n) \neq \emptyset \Rightarrow \exists i\colon \operatorname{Int}(A_i) \neq \emptyset$$

Proof (46) Let $U_0 := U(x_0, \varepsilon_0) \subset \bigcup_{n=1}^{\infty} A_n$ for some $\varepsilon_0 > 0$. Now assume that all the closed sets A_n are nowhere dense:

$$\forall \varepsilon > 0, x \in X, n \in \mathbb{N}\colon (X \setminus A_n) \cap U(x, \varepsilon) \neq \emptyset$$

Then $(X \setminus A_1) \cap U_0$ is open and not empty. So there exists a closed Ball $K_1 = K(x_1, \varepsilon_1), 0 < \varepsilon_1 < \min(1, \varepsilon_0)$ with $K_1 \subset (X \setminus A_1) \cap U_0$ and $(X \setminus A_2) \cap U(x_1, \varepsilon_1)$ is open and not empty.

We repeat this step:

$$\exists K_n := K(x_n, \varepsilon_n), 0 < \varepsilon_n < \min\left(\frac{\varepsilon_{n-1}}{2}, 2^{-(n-1)}\right)$$
$$K_n \subset (X \setminus A_n) \cap U(x_{n-1}, \varepsilon_{n-1}) \subset K_{n-1}$$

According to Lemma 45 (Conditions for Nowhere Density) there is a unique x, $\{x\} = \bigcap_{n \in N} K_n$. So $x \in \bigcap_{n=1}^{\infty} (X \setminus A_n) = X \setminus \bigcup_{n=1}^{\infty} A_n \Rightarrow x \notin \bigcup_{n=1}^{\infty} A_i$. On the other hand $x \in K_1 \subset U_0 \subset \bigcup_{n=1}^{\infty} A_i$ which is a contradiction. So not all A_n can be nowhere dense, which means one has to have an nonempty interior.

47 **Corollary: Banach-Steinhaus Theorem, Uniform-Boundedness Principle**

Let $(V, \|\cdot\|)$ be a Banach space, $(W, \|\cdot\|)$ a normed space and $Z \subset \mathcal{B}(V, W)$ a subset such that $\{\, \|T(v)\| \mid T \in Z \,\}$ is bounded for each $v \in V$. Then there exists a common bound $C < \infty$ with $\forall T \in Z\colon \|T\| < C$.

Proof (47) The following set is closed as intersection of closed sets:

$$A_n := \left\{\, v \in V \;\middle|\; \forall T \in Z\colon \|T(v)\|_W \leq n \,\right\} = \bigcap_{T \in Z} \{\, v \in V \mid \|T(v)\| \leq n \,\}$$

So according to Theorem 46 (Special Baire Category Theorem) there are $n_0 \in \mathbb{N}, 0 < \varepsilon < 1, v_0 \in V$ such that

$$U(v_0, \varepsilon) = \left\{ \varepsilon v + v_0 \;\middle|\; \|v\| < 1 \right\} \subset A_{n_0}$$
$$\Rightarrow \forall T \in Z, v \in V, \|v\| < 1 \colon \|T(\varepsilon v + v_0)\| \le n_0$$
$$\Rightarrow \|T(v)\| \le \frac{1}{\varepsilon}(\|T(\varepsilon v) + v_0\| + \|T(v_0)\|) \le \frac{3n_0}{\varepsilon}$$
$$\Rightarrow \sup_{T \in Z} \|T\| \le C := \frac{3n_0}{\varepsilon} < \infty$$

48 **Remark: Version of Theorem 46 (Special Baire Category Theorem)**

We will only use the following version of Theorem 46 (Special Baire Category Theorem):
Let (X, ρ) be a complete metric space and (A_i) an ascending sequence of closed subsets of X $(A_1 \subset A_2 \subset \dots)$ which covers X (i.e. $X = \bigcup_{n\in\mathbb{N}} A_n$). Then

$$\exists n_0 \in \mathbb{N}, x_0 \in X, \varepsilon > 0 \colon U(x_0, \varepsilon) \subset A_{n_0}$$

5 Open Mapping Theorem

49 **Lemma: Closure Irrelevance**

Let V be a Banach space, W a normed vector space , $T \colon V \to W$ a bounded linear map and $U_W(0, \varepsilon) \subset \overline{T(U_V(0,1))}$ for some $\varepsilon > 0$. Then $U_W(0, \varepsilon) \subset T(U_V(0,1))$.

Proof (49) Let $y \in U_W(0, \varepsilon) = \varepsilon U_W(0,1)$ (ie $\|y\| < \varepsilon$) then we can find $\|y\| < \varepsilon_0 < \varepsilon$ and $\alpha \ge 0$ with $\frac{\varepsilon_0}{\varepsilon} + \alpha < 1$ so that $\|\frac{\varepsilon_0}{\varepsilon} y\| < \varepsilon_0 < \varepsilon$. By induction we can show the existence of $(x_i)_{i\in\mathbb{N}}, x_i \in U_V(0,1)$ with

$$|T(\sum_{k=0}^{n} \alpha^k x_k) - \frac{\varepsilon}{\varepsilon_0} y| < \alpha^{n+1} \varepsilon_0$$

- $n = 0$: $\|\bar{y}\| < \varepsilon_0$ for $\bar{y} := \frac{\varepsilon_0}{\varepsilon} y$, $\frac{\varepsilon_0}{\varepsilon} \left(\frac{1}{1-\alpha}\right) < 1$. We find $y_0 \in T(U_V(0,1))$ with $\|\bar{y} - y_0\| < \alpha\varepsilon_0$. Thus there exists $x_0 \in U_V(0,1)$ with $T(x_0) = y_0$.
- $n \Rightarrow n+1$: Suppose we have found $x_0, \dots, x_n \in U_V(0,1)$ with

$$\|\bar{y} - T(\sum_{k=0}^{n} \alpha^k x_k)\| < \alpha^{n+1} \varepsilon_0$$
$$\bar{\bar{y}} := \frac{1}{\alpha^{n+1}} \left(\bar{y} - T(\sum_{k=0}^{n} \alpha^k x_k) \right)$$

then $\|\bar{\bar{y}}\| < \varepsilon_0$. Therefore we find $y_{n+1} \in T(U_V(0,1))$ with $\|\bar{\bar{y}} - y_{n+1}\| < \alpha^{n+2}\varepsilon_0$

$$\Rightarrow \exists x_{n+1} \in U_V(0,1)\colon y_{n+1} = T(x_{n+1})$$
$$\Rightarrow \|\bar{\bar{y}} - T(x_{n+1})\| < \alpha^{n+2}\varepsilon_0$$
$$\Rightarrow \|\bar{y} - T(\sum_{k=0}^{n+1} \alpha^k x_k)\| < \alpha^{2n+3}\varepsilon_0 < \varepsilon^{n+2}\varepsilon_0$$

$\sum_{k=0}^{\infty} \alpha^k x_k$ is absolutely convergent because

$$\sum_{k=0}^{\infty} \|\alpha^k x_k\| < \sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha}$$
$$\Rightarrow \bar{x} := \sum_{k=0}^{\infty} \alpha^k x_k$$

exists by completeness of V.

$$T(\bar{x}) = \bar{y} \Rightarrow T(\frac{\varepsilon}{\varepsilon_0}\bar{x}) = y$$

$$\|\frac{\varepsilon}{\varepsilon_0}\bar{x}\| = \frac{\varepsilon}{\varepsilon_0} \lim_{n\to\infty} \|\sum_{k=0}^{n} \alpha^k x_k\| \leq \frac{\varepsilon}{\varepsilon_0} \lim_{n\to\infty} \sum_{k=0}^{n} \alpha^k \|x_k\| = \frac{\varepsilon}{\varepsilon_0} \sum_{k=0}^{\infty} \alpha^k \|x_k\|$$
$$< \frac{\varepsilon}{\varepsilon_0} \sum_{k=0}^{\infty} \alpha^k = \frac{\varepsilon_0}{\varepsilon}\left(\frac{1}{1-\alpha}\right)$$

because $\|x_k\| < 1$.

50 **Definition: "Open Map"**

A map $f\colon X \to Y$ between topological space X and Y is called *open* if all the images of open sets in X are open in Y.

51 **Remark: Condition for Openness**

A map $T \in \mathcal{B}(V, W)$ is open iff

$$\exists \varepsilon > 0\colon U_W(0,\varepsilon) \subset T(U_V(0,1))$$

52 **Theorem: Open Mapping Theorem (Banach-Schauder)**

Let V, W both be Banach Spaces and $T \in \mathcal{B}(V, W)$ surjective. Then T is open.

Proof (52) Let $A_n = \overline{T(nU_V(0,1))}$. These A_n are closed subsets with $A_1 \subset A_2 \subset \ldots$ and $\bigcup_{n\in\mathbb{N}} A_n \supset T(\bigcup_{n\in\mathbb{N}} nU_V(0,1)) = T(V) = W$. So Special Baire Category Theorem applies and we find $y \in W$, $\varepsilon_1 > 0$ and $n_0 \in \mathbb{N}$ such that

$$U_W(y,\varepsilon) \subset A_{n_0} = \overline{T(u_0 U_V(0,1))} = \overline{n_0 T(U_V(0,1))} = n_0 \overline{T(U_V(0,1))}$$
$$\frac{1}{n_0} y + \frac{\varepsilon_1}{n_0} U_W(0,1) \subset \overline{T(U_V(0,1))}$$
$$\varepsilon := \frac{\varepsilon_1}{2n_0}$$
$$\tfrac{1}{2}\left(\frac{1}{n_0} y + \frac{\varepsilon_1}{u_0} U_W(0,1)\right) - \tfrac{1}{2}\left(\frac{1}{n_0} y + \frac{\varepsilon_1}{n_0} U_W(0,1)\right)$$
$$\subset \tfrac{1}{2}\left(\overline{T(U_V(0,1))} - \overline{T(U_V(0,1))}\right) \subset \overline{T(U_V(0,1))}$$
$$(x,y) \in V \times V \mapsto x + y \in V \text{ is continuous}$$
$$x \in V \mapsto -x \in V \text{ is continuous}$$

(Note that we are using arithmetic operations on sets as elementwise operations.) So by the convexity of $U_V(0,1)$ we get

$$\tfrac{1}{2} U_V(0,1) - \tfrac{1}{2} U_V(0,1) \subset U_V(0,1)$$
$$\Rightarrow \tfrac{1}{2}\left(T(U_V(0,1)) - T(U_V(0,1))\right) \subset T(U_V(0,1)) \subset \overline{T(U_V(0,1))}$$

In general for arbitrary $X, Y \subset W$ we have $\bar{X} - \bar{Y} \subset \overline{X - Y}$.

$$\Rightarrow \tfrac{1}{2}\left(\overline{T(U_V(0,1))} - \overline{T(U_V(0,1))}\right) \subset \overline{T(U_V(0,1))}$$
$$\varepsilon U_W(0,1) \subset \varepsilon\left(U_W(0,1) - U_W(0,1)\right)$$
$$\Rightarrow \varepsilon U_W(0,1) \subset \overline{T(U_V(0,1))}$$
$$\Rightarrow \varepsilon U_W(0,1) \subset T(U_V(0,1))$$

which is what we wanted to prove.

53 **Corollary: Inverse Mapping Theorem**

If V, W are Banach spaces and $T\colon V \to W$ is a linear, continuous and bijective map then $T^{-1}\colon W \to V$ is continuous.

Proof (53) We need to show $T^{-1}(U_W(0,\varepsilon)) \subset U_V(0,1)$ for some $\varepsilon > 0$. According to Theorem 52 (Open Mapping Theorem (Banach-Schauder)) we get $T^{-1}(U_W(0,1)) \subset \frac{1}{\varepsilon} U_V(0,1) \Leftrightarrow \|T^{-1}\| \leq \frac{1}{\varepsilon} < \infty$ which is what we need because of linearity / homogeneity.

54 **Definition: "Compatible Norms"**

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on a vector space V. We say $\|\cdot\|_2$ *majorizes* $\|\cdot\|_1$ if there is a $C < \infty$ such that

$$\forall v \in V\colon \|v\|_1 \leq C\|v\|_2$$

Two norms are called *compatible* if either one majorizes the other and *equivalent* if they majorize each other.

55 **Corollary: Equivalence Condition**

Compatible norms on a Banach space are always equivalent.

Proof (55) Follows according to Corollary 53 (Inverse Mapping Theorem) with $T: T(v) = v$.

56 **Remark: Sum Space**

Let $(V, \|\cdot\|), (W, \|\cdot\|)$ be normed metric spaces.

1. It is easy to check, that $(V \oplus_1 W) = (V \times W, \|\cdot\|_1)$ with $\|(v, w)\|_1 := \|v\| + \|w\|$ is a normed space.
2. The canonical projections form $V \oplus_1 W$ to V and W are isometric linear maps.
3. $V \oplus_1 W$ is a Banach space iff V and W are Banach spaces.

57 **Corollary: Equivalence of Complete Norms**

Let V, W be Banach spaces, then all the complete norms on $V \oplus W$ whose canonical projections are continuous, are equivalent. In particular they are equivalent to the above defined norm $\|\cdot\|_1$.

Proof (57) There are $C_1, C_2 < \infty$, e.g. $C_1 := \|\pi_V\|$ and $C_2 := \|\pi_W\|$, such that $\|(x, 0)\| \leq C_1\|x\|$ and $\|(0, y)\| \leq C_2\|y\|$ for any $x \in V, y \in W$. That gives us

$$\|(x, y)\| \leq \|(x, 0)\| + \|(0, y)\| \leq \max(C_1, C_2)(\|x\| + \|y\|)$$

The function $T, T(x, y) := (x, y)$ is a continuous bijective map $V \oplus_1 W \to (V \oplus W, \|\cdot\|)$. So T is invertible by Corollary 53 (Inverse Mapping Theorem).

58 **Definition: "Graph"**

The *graph* of a map $f: X \to Y$ is the set $\mathcal{G}(f) := \{ (x, f(x)) \in X \times Y \mid x \in X \}$.

59 **Lemma: Linear Graph**

For V, W vector spaces and $f: V \to W$ the Graph $\mathcal{G}(f)$ is a linear subspace of $V \oplus W$ iff f is a linear map.

60 **Theorem: Closed Graph Theorem**

Let V, W be Banach spaces and $T: V \to W$ a linear map, then T is continuous iff $\mathcal{G}(T)$ is closed in $V \oplus W$.

Proof (60) Define $S: V \to V \oplus W, S(x) := (x, T(x))$ (i.e. $\pi_2 \circ S = T$). S is a bijection between V and $\mathcal{G}(T)$ and the inverse map is $\pi_1|_{\mathcal{G}(T)}$.

Assume T is continuous and $(x_n, T(x_n))$ is a sequence in $\mathcal{G}(T)$ which converges to (x, y). Because T is continuous $y = T(x)$ and therefore $\mathcal{G}(T)$ is closed.

Now assume $\mathcal{G}(T)$ is closed. Since $V \oplus_1 W$ is Banach, $\mathcal{G}(T)$ is Banach. So the inverse S of $\pi_1|_{\mathcal{G}(T)}$ is continuous by Corollary 53 (Inverse Mapping Theorem) because π_1 is continuous. That means $T = \pi_2 \circ S$ is continuous as the composition of two continuous functions.

6 Hahn-Banach Extension Principle

61 **Definition: "Sublinearity"**

Let V be a real vector space and $p\colon V \to \mathbb{R}$ then p is called *sublinear* if

1. $\forall \lambda \geq 0\colon p(\lambda x) = \lambda p(x)$
2. $p(x+y) \leq p(x) + p(y)$

For example all linear maps are sublinear.

62 **Theorem: Hahn-Banach Extension Theorem**

Let V be a vector space over $\mathbb{R}$ and $p\colon V \to \mathbb{R}$ a sublinear functional. Suppose that $L \subset V$ is a linear subspace of V and $f\colon L \to \mathbb{R}$ is a linear map with $\forall v \in L\colon f(v) \leq p(v)$.

Then there exists a linear map $g\colon V \to \mathbb{R}$ with

$$\begin{aligned} &\forall v \in L\colon g(v) = f(v) \\ &\forall v \in V\colon g(v) \leq p(v) \end{aligned}$$

Proof (62) It is possible to give a proof using Zorns Lemma, but we want to do it more constructive.

Notation: we write (and already wrote) $X - Y$ for $\{\, x - y \mid x \in X, y \in Y \,\}$ which is not the same as $X \setminus Y$!

If $L = V$ we do not have to prove anything. Let $x_0 \in V \setminus L$ and attempt to find an extension $g\colon L + \mathbb{R}x_0 \to \mathbb{R}$ with $g|_L = f$ and $g \leq p|_{(L+\mathbb{R}x_0)}$. We have to find a fitting value $\gamma \in \mathbb{R}$ such that $g(v + \alpha x_0) := f(v) + \alpha\gamma$ satisfies the above conditions.

$$\begin{aligned} &g(v + \alpha x_0) = f(v) + \alpha\gamma \leq p(v + \alpha x_0) \\ &\forall \alpha \in \mathbb{R}, \alpha \neq 0, v \in L\colon \alpha\gamma \leq p(v + \alpha x_0) - f(v) \end{aligned} \tag{6-1}$$

So we calculate:

$$\begin{aligned} &\forall x, y \in L\colon f(x) + f(y) = f(x+y) \leq p(x+y) \leq p(x + x_0) + p(y - x_0) \\ &\Rightarrow \forall x, y \in L\colon f(y) - p(y - x_0) \leq p(x + x_0) - f(x) \\ &\Leftrightarrow \underbrace{\sup_{z \in L} f(z) - p(z - x_0)}_{=:\gamma_1} \leq \underbrace{\inf_{z \in L} p(z + x_0) - f(z)}_{=:\gamma_2} \end{aligned}$$

We will show that Equation 6-1 holds: Let $\gamma \in [\gamma_1, \gamma_2]$ and $x \in L$ be fixed. Remember $\alpha \neq 0$, so for $\alpha > 0$ we get

$$z := \frac{1}{\alpha} x \Rightarrow \alpha\gamma \leq \alpha\gamma_2 \leq \alpha p(z + x_0) - \alpha f(z) = p(x + \alpha x_0) - f(x)$$

and for $\alpha < 0$ we have

$$z := \frac{1}{|\alpha|}x = -\frac{1}{\alpha}x \Rightarrow f(zr - p(z - x_0) \leq \gamma_1 \leq \gamma$$
$$\Rightarrow -\gamma \leq \frac{1}{|\lambda|}p(x + \alpha x_0) - \frac{1}{|\alpha|}f(x)$$
$$\Rightarrow \alpha\gamma \leq p(x + \alpha x_0) - f(x)$$

So now we have found an extension $g, g(x + \alpha x_0) = f(x) + \alpha\gamma$ which satisfies the conditions $g|_L = f$ and $\forall v \in L_0 : g \leq p$ with $L_0 := L + \mathbb{R}x_0$.

Forthwith we can do the following: Suppose V/L has a countable vector space basis $\{x_0 + L, x_1 + L, \ldots\}$. Then we concider the linear subspaces

$$L_n := L + \mathbb{R}x_0 + \mathbb{R}x_1 + \ldots + \mathbb{R}x_n$$

and can find linear maps $g_n : L_n \to \mathbb{R}$ with $g_{n+1}\big|_{L_n} = g_n \Rightarrow g_n\big|_L = f$ and $g_n \leq p$ on L_n by inductively applying the above argument. Because V/L had a countable basis, we have $V = \bigcup_{n=0}^{\infty} L_n$ and we can define $g : V \to \mathbb{R}$ by $g(x) := g_n(x)$ for $x \in L_n$ which satisfies the condition of the theorem in this case.

Up till now we have not used the Axiom of Choice. But to get the most general situation as used in the theorem we have to use Zorns Lemma.

General Case Let $X \subset \wp(V) \times \wp(X \times \mathbb{R}) = 2^V \times 2^{(V \times \mathbb{R})}$ denote the set of all pairs (W, g) consisting of a linear subspace $W \subset V$ with $L \subset W$ and $g : W \to \mathbb{R}$ a linear map with $g|_L = f$ and $g \leq p$ on W. In particular we have $(L, f) \in X$. Then we can introduce a partial order on X as

$$(W_1, g_1) \prec (W_2, g_2) :\Leftrightarrow W_1 \subset W_2 \wedge \forall v \in W_1 : g_2(v) = g_1(v)$$

It is easy to check that $\prec$ is a well-defined partial order on X as it satisfies reflexivity, antisymmetry and transitivity.

$\prec$ satisfies the preconditions of Zorns Lemma Let $Z \subset X$ be a linearly ordered subset, then we get

$$(W_1, g_1), (W_2, g_2) \in Z$$
$$\Rightarrow (W_1 \subset W_2 \wedge \forall v \in W_1 : g_2(v) = g_1(v))$$
$$\vee (W_2 \subset W_1 \wedge \forall v \in W_2 : g_1(v) = g_2(v)$$

This implies that $W_Z := \bigcup_{(W,g) \in Z} W$ is a linear subspace of V and for any $x, y \in W_2$ there exists some $(W, g) \in Z$ with $x, y \in W$. Now

$$\tilde{g}(x) := g(x) \Leftrightarrow x \in W, (W, g) \in Z$$

is well-defined and linear. So from the definition of Z and $\tilde{g}$ we immediately get $\tilde{g}|_L = f$ and $\forall v \in W_2 : \tilde{g}(v) \leq p(v)$. Furthermore $\tilde{g}|_W = g$ and $W \subset W_2$ for all $(W, g) \in Z$ so that we know

$(W_2, \tilde{g}) \in X$ is the upper bound we were looking for: $(W, g) \prec (W_2, \tilde{g})$. So now we have a maximal Element $(W, g) \in X$ according to Zorns Lemma. Suppose that $W \neq V$. Then there is some $x_0 \in V \setminus W$ and we can extend g to $g^*: W + \mathbb{R}x_0 \to \mathbb{R}$ with $g^*\big|_W = g$ and $g^* \leq p$ on $W + \mathbb{R}x_0$. This means $(W, g) \prec (W + \mathbb{R}x_0, g^*)$ which is a contradiction as (W, g) was maximal. So g is the extension of f we were looking for.

63 Axiom: Axiom of Choice

Let $X, Y \neq \emptyset$ be sets and $f: X \hookrightarrow Y$ surjective. Then there exists a section of f, i.e. a map $s: Y \to X$ such that $f \circ s = \mathrm{id}_Y$.

64 Theorem: Zorns Lemma

Let $(X, \leq)$ be a partially ordered, non-empty set and assume that for every linearly ordered subset $Z \subset X$ (i.e. $x, y \in Z \Rightarrow x \leq y \vee y \leq x$) there is an upper bound ($w_Z \in X: \forall x \in z: x \leq w_Z$). Then there is a maximal Element in X ($w \in X: \forall x \in X: w \leq x \Rightarrow w = x$).

Zorns Lemma is equivalent to the Axiom of Choice in ZF.

In the following we are going to explore some consequences of Theorem 62 (Hahn-Banach Extension Theorem).

65 Theorem: Extension on Seminormed Spaces

Let $(V, \|\cdot\|)$ be a seminormed $\mathbb{R}$-Vectorspace. If $L \subset V$ is a linear subspace and $f: L \to \mathbb{R}$ is a linear functional, $0 < C < \infty$ so that $\forall x \in L: |f(x)| \leq C\|x\|$ (in other words $\|f\| \leq C$) then there exists a linear functional $g: V \to \mathbb{R}$ with $\|g\| \leq C$ and $g|_L = f$.

Proof (65) Take $p(x) := C\|x\|$ and notice that $p(hx) = hp(x), h \geq 0$ and $f(x) \leq |f(x)|$. Then the theorem follows by applying Theorem 62 (Hahn-Banach Extension Theorem) to f, p, L.

66 Corollary: Complex Version of Extension on Seminormed Spaces

Let $(V, \|\cdot\|)$ be a seminormed $\mathbb{C}$-vector space, $L \subset V$ a $\mathbb{C}$-linear subspace, $f: L \to \mathbb{C}$, C-linear and $0 < C_0 < \infty$ with $\forall x \in L: |f(x)| \leq C_0\|x\|$. Then there is a linear $g: V \to \mathbb{C}$ with $\|g\| \leq C_0$ and $g|_L = f$.

67 Corollary: Extension on Normed Spaces

Let V be a normed $\mathbb{K}$-vector space, $L \subset V$ a linear subspace and $f: L \to \mathbb{K}$ linear and continuous. Then there is a $F: V \to \mathbb{K}$ such that $\|F\| = \|f\|$ and $F|_L = f$.

68 Corollary: Hahn-Banach Separation

Let $V \neq \{0\}$ be a normed $\mathbb{K}$-vector space. For each $x \in V$ there is a $f \in V' = \mathcal{B}(V, \mathbb{K})$ with $f(x) = \|x\|$ and $\|f\| = 1$.

Proof (68) For one-dimensional vector spaces the statement is obvious. In the other cases we only have to extend $\mathbb{R}y$ with $x \in \mathbb{R}y$ for suitable $y \in Y \setminus \{0\}$ to the whole V according to Corollary 67

69 **Remark: Natural, Isometric Embedding**

In particular Corollary 68 implies, that the natural map

$$e_V: V \to V'' = \mathcal{B}(\mathcal{B}(V, \mathbb{K}), \mathbb{K}), e_V(x)(f) := f(x), x \in V, f \in V'$$

is an isometric linear map.

70 **Theorem: Stronger Extension**

Let V be a real vector space (not necessarily normed), $M \subset V, M \neq \emptyset$ a convex subset and $p: V \to \mathbb{R}$ a sublinear functional. Then there exists a linear map $f: V \to \mathbb{R}$ with $f \leq p$ and $\inf_{x \in M} p(x) = \inf_{x \in M} f(x)$.

Proof (70) We have $p(0) = 0$, $g(0) = 0$ and $L = \{0\}$ so according to Theorem 62 (Hahn-Banach Extension Theorem) we have a $f: V \to \mathbb{R}$ with $f \leq p$ and $f(0) = g(0) = 0$. So if $-\infty = \inf_{x \in M} p(x)$ each $f: V \to \mathbb{R}$ with $f \leq p$ already is a solution. Otherwise let $m := \inf_{x \in M} p(x)$ and

$$\tilde{p}(x) := \inf \{ p(x + ty) - tm \mid y \in M, t \geq 0 \}$$

Now we get $p(x+ty) - tm > -p(-x)$ by sublinearity and definition of m. Thus $\tilde{p}(x) \geq -p(-x)$ so that $\tilde{p}$ is well-defined. Clearly $\tilde{p}(0) = 0$ and $\tilde{p}$ is positive homogeneous ($\lambda > 0$: $\lambda \tilde{p}(x) = \tilde{p}(\lambda x)$). The convexity of M implies with $t_1 > 0, t_2 > 0$ by taking the infimum on the right side:

$$\tilde{p}(x + z) \leq p(x + z + (t_1 + t_2)y_3) - (t_1 + t_2)m$$
$$\leq (p(x + t_1 y_1) - t_1 m) + (p(z + t_2 y_2) - t_1 m)$$
$$y_3 := \frac{t_1}{t_1 + t_2} y_1 + \frac{t_2}{t_1 + t_2} y_2$$
$$\Rightarrow \tilde{p}(x + z) \leq \tilde{p}(x) + \tilde{p}(z)$$

So by Theorem 62 (Hahn-Banach Extension Theorem) applied to $0 \leq \tilde{p}(0) = 0$ on $L = \{0\}$ there is a linear $f: V \to \mathbb{R}$ with $f \leq \tilde{p}$. This means $f \leq p$ and so for $x \in M$ we have $-f(x) = f(-x) \leq \tilde{p}(-x) \leq p(-x + x) - m = -m \Rightarrow \forall x \in M: m \leq f(x)$.

71 **Corollary: Separation Theorem (Banach-Mazur)**

Let V be a real normed vector space and A, B non-empty convex subsets of X with positive distance:

$$\operatorname{dist}(A, B) = \inf_{a \in A, b \in B} \|a - b\| > 0$$

Then there is a linear continuous map $f: V \to \mathbb{R}$ with $f(A) \cap f(B) = \emptyset$. In fact there is even a f with $d(f(A), f(B)) = d(A, B)$ and $\|f\| = 1$.

Proof (71) We know that $A - B$ is convex. Thus there is a f with $f \leq \|\cdot\|$ ($\Rightarrow \|f\| \leq 1$) such that

$$0 < d(A, B) = \inf_{(a-b) \in A-B} \|a - b\| = \inf_{x \in A-B} f(x) = \inf_{a \in A} f(a) - \sup_{b \in B} f(b)$$

72 **Corollary: Open Cones**

Let $(V, \|\cdot\|)$ be a normed $\mathbb{R}$-vector space. If $C \subset V$ is open, convex and satisfies

$$\forall \lambda > 0 \colon \lambda C \subset C$$

(i.e. it is an open *cone*) with $0 \notin C \neq \emptyset$. Then

1. For all $a \in V \setminus \bar{C}$ there is a $f \in V'$ with $f(C) \subset \mathbb{R}_+$ and $f(a) < 0$.
2. For every $c_0 \in C$ there exists $f \colon V \to \mathbb{R}$ with $f(C) \subset \mathbb{R}_+$ and $f(c_0) > 0$.
3. Let $\hat{C} := \left\{ f \in V' \mid f(C) \subset \mathbb{R}_+ \right\}$ Then the closure $\bar{C}$ of C is given by
$$\bar{C} = \left\{ x \in V \mid \forall f \in \hat{C} \colon f(x) \geq 0 \right\}$$
4. Each linear map $f \colon V \to \mathbb{R}$ with $f(C) \subset \mathbb{R}_+$ is automatically continuous.

Proof (72)

1. If $a \notin \bar{C}$ then there is some $\varepsilon > 0$ with $a + U(0, \varepsilon) = U(a, \varepsilon) \subset V \setminus \bar{C}$ and we get
$$d(\{a\}, C) \geq \varepsilon > 0, \quad f \neq 0$$
So according to Corollary 71 (Separation Theorem (Banach-Mazur)) there is some linear and continuous $f \colon V \to \mathbb{R}$ with $d(f(a), f(C)) > \varepsilon$. But $f(C)$ is a cone in $\mathbb{R}$ and $f(C) \subset \mathbb{R}_+$. Since C is open, the vector subspace $C - C = \bigcup_{d \in C} C - d$ is an open vectorspace too. That means $C - C = V$ and $f(C) - f(C) = \mathbb{R}$. so there are only two cases: $f(C) = (0, \infty)$ and $f(C) = [0, \infty)$. Anyway $f(a) < 0$.
2. Because the first proposition was already shown it is sufficient to show that $-c \notin C$ if $c \in C$.
Since C is open, there is some $\varepsilon > 0$ with $U(c, \varepsilon) \subset C$. Suppose now that $-c \in \bar{C}$. Then
$$\exists v \in (-U(c, \varepsilon)) \cap C \colon v = -c - w_1 \in C$$
$$\Rightarrow 0 = \tfrac{1}{2}\big((-c - w_1) + (c + w_1)\big) \in C$$
So now the first proposition applies to $-c$.
3. $f^{-1}(\mathbb{R}_+) \subset V$ is closed by continuity of $f \in \hat{C}$ and by our definition of $\hat{C}$ we get $C \subset f^{-1}(\mathbb{R}_+)$ and
$$\left\{ x \in V \mid \forall f \in \hat{C} \colon f(x) \geq 0 \right\} = \bigcap_{f \in \hat{C}} f^{-1}(\mathbb{R}_+) \quad \text{is closed}$$
$$\Rightarrow \bar{C} \subset \left\{ x \in V \mid \forall f \in \hat{C} \colon f(x) \geq 0 \right\} \quad \text{is closed}$$
If $x \notin \bar{C}$ we can find a $f \in \hat{C}$ with $f(x) < 0$ by the first proposition. So
$$\bar{C} = \left\{ x \in V \mid \forall f \in \hat{C} \colon f(x) \geq c \right\}$$
and we have the following Formular:
$$\hat{\hat{C}} \cap e_V(V) = C$$

4. We have

$$f^{-1}(0) = V \setminus \left(f^{-1}((0,\infty)) \cup f^{-1}((-\infty,0)) \right)$$

If $f \neq 0$ then $f(C) \supseteq (0,\infty)$ because $C - C = V$. And so $V \setminus f^{-1}([0,\infty))$ contains an open subset, namely $-C$. That means $f^{-1}(0)$ must be closed.

7 Exactness, Dual spaces and Reflexivity

7.1 Quotient Spaces

Let $(V, \|\cdot\|)$ be a seminormed vector space and $L \subset V$ a linear subspace. Then the quotient space of V through L is seminormed by

$$Q = V/L = \{ [v] := v + L \mid v \in V \}$$
$$\|[v]\|_Q := \inf_{x \in L} \|v + x\|$$

and $\pi_L : V \to V/L, \pi_L(v) = v + L$ is the natural projection.

73 **Lemma: Basic Properties of Quotient Spaces**
In this situation

1. $(V/L, \|\cdot\|_Q)$ is a seminormed space.
2. $(V/L, \|\cdot\|_Q)$ is normed iff L is closed in V.
3. If $(V, \|\cdot\|)$ and $(V/L, \|\cdot\|_Q)$ are normed, $(V, \|\cdot\|)$ is Banach iff L and V/L are both Banach spaces.

Proof (73)

2. If V/L is separated (i.e. $\|\cdot\|_Q$ is a norm) and $x \in \bar{L}$ there is a sequence $(x_n)_{n\in\mathbb{N}}, x_n \in L$ with $0 \leq \|x + L\|_Q \leq \|x - x_n\|_Q \to 0$. So according to the definition of $\|\cdot\|_Q$ we get $\|x + L\|_Q = 0$ so that $x + L = 0 + L \Rightarrow x \in L$.

   If, on the other hand, L is closed in V, then $\|x + L\|_Q = 0$ gives us the existence of some sequence $(x_n)_{n\in\mathbb{N}}, x_n \in L$ with $\lim_{n\to\infty} \|x - x_n\| = \|x + L\|_Q = 0$ and so $x \in \bar{L}$ with respect to the metric on V.

3. Let $(V, \|\cdot\|)$ be Banach and $(V/L, \|\cdot\|_Q)$ be normed. We will show that L is closed in V and thus Banach. Suppose $(x_n + L)_{n\in\mathbb{N}}, x_n \in V$ is Cauchy in V/L, then by induction we select certain $y_n \in V$ with the following properties:

$$y_1 := x_1$$
$$y_n \in x_n + L \quad \wedge \quad \|y_{n+1} - y_n\| \leq \|(x_{n+1} + L) - (x_n + L)\|_Q + 2^{-(n+1)}$$

   So $(x_n + L)_{n\in\mathbb{N}}$ being Cauchy in V/L implies y_n is Cauchy in V. So y_n converges to some y and we claim that $\lim_{n\to\infty} y_n + L = y + L$.

74 **Definition: "Short Exact Sequence"**

A *short exact sequence* (in the algebraic sense) is a chain of linear maps $T_1\colon V_1 \to V_2, T_2\colon V_2 \to V_3$ so that $\mathrm{Ker}(T_1) = \{0\}$, $\mathrm{Im}(T_1) = \mathrm{Ker}(T_2)$ and $\mathrm{Im}(T_2) = V_3$.

Algebraic rule for short exact sequences Consider the following net of 6 sequences:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U_1 & \longrightarrow & U_2 & \longrightarrow & U_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_1 & \longrightarrow & V_2 & \longrightarrow & V_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W_1 & \longrightarrow & W_2 & \longrightarrow & W_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Then it is sufficient to show that 5 of them are short exact, for all 6 of them to be short exact. This is called the 5 of 6 lemma or the 3 times 3 lemma.

Application

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_1 & \xrightarrow{\text{injectiv}} & V_2 & \xrightarrow{\text{surjectiv}} & V_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\text{injectiv}} & & \downarrow{\scriptstyle\text{bijectiv}} & & \downarrow{\scriptstyle\text{injectiv}} & & \\
0 & \longrightarrow & W_1 & \xrightarrow{\text{injectiv}} & W_2 & \xrightarrow{\text{surjectiv}} & W_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W_1/V_1 & \longrightarrow & W_2/V_2 & \longrightarrow & W_3/V_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

If the two first horizontal sequences are short exact, $W_1 = V_1$ and $W_3 = V_1$ or the downward maps are bijectiv.

75 **Definition: "Metrically Short Exact Sequence"**

Let V_1, V_2, V_3 be Banach spaces and $T_1 \in \mathcal{B}(V_1, V_2)$, $T_2 \in \mathcal{B}(V_2, V_3)$. Then

$$0 \longrightarrow V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} V_3 \longrightarrow 0$$

is a *short exact* in a metrical sense if T_1 is isometric, $T_1(V_1) = \operatorname{Ker}(T_2)$ and $[T_2]\colon V_2/\operatorname{Ker}(T_2) \to V_3$ is an isometric isomorphism onto V_3.

76 **Remark: Prototype**

Let V be Banach and $L \subset V$ a closed linear subspace then $\pi_L\colon V \to V/L$ maps the open unit Ball of V onto the open unit ball of V/L. The sequence $0 \to L \hookrightarrow V \xrightarrow{\pi_L} V/L \to 0$ is the prototype of metrically short exact.

A sequence is metrically short exact iff T_1 is isometric, $\operatorname{Ker}(T_2) = T_1(V_1)$ and T_2 maps the open unit ball of V_2 onto the open unit ball of V_3.

There is a unique isometric isomorphism S so that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V_1 & \xrightarrow{T_1} & V_2 & \xrightarrow{T_2} & V_3 & \longrightarrow & 0 \\
 & & \downarrow S & & & & \downarrow S & & \\
0 & \longrightarrow & \operatorname{Ker} T_2 & \longrightarrow & V_2 & \longrightarrow & V_2/\operatorname{Ker}(T_2) & \longrightarrow & 0
\end{array}$$

Recall The dual space was defined as $V' = \mathcal{B}(V, \mathbb{K})$ with the operator norm $\|\cdot\|_{\mathcal{B}}$. And for $T\colon V \to W$ the map $T'\colon W' \to V'$ is defined as $T'(f) = f \circ T$.

Then the above comutativ diagram can be observed with $T_1 = \varepsilon_L\colon L \to V$, $T_2 = \pi_L\colon V \to V/L$ and the orthogonal space $L^\perp := \left\{ f \in \mathcal{B}(V, \mathbb{K}) \;\middle|\; f|_L = 0 \right\}$ in the dual spaces:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (V/L)' & \xrightarrow{\pi'_L} & V' & \xrightarrow{\varepsilon'_L} & L' & \longrightarrow & 0 \\
 & & \downarrow S & & & & \downarrow S & & \\
0 & \longrightarrow & L^\perp = \operatorname{Ker} \varepsilon'_L & \longrightarrow & V' & \xrightarrow{\pi_{L^\perp}} & V'/L^\perp & \longrightarrow & 0
\end{array}$$

which follows from Theorem 62 (Hahn-Banach Extension Theorem) applied to $f \in K_{L'}(0, 1)$.

There is a natural linear map $e_V\colon V \to V'' = \mathcal{B}(\mathcal{B}(V, \mathbb{K}), \mathbb{K})$ as $e_V(v)(f) := f(v)$. And we know that $\|e_V(v)\|_{V''} \leq \|v\|$ for any seminorm $\|\cdot\|$ on V. If $(V, \|\cdot\|)$ is normed we even get $\|e_V(v)\| = \|v\|$.

Proof (69) Consider the linear subspace $(\mathbb{K}v, \|\cdot\|)$ which is isomorphic to $(\mathbb{K}, |\cdot|)$ with the linear isometry $f\colon \alpha v \mapsto \alpha\|v\|$. So according to Theorem 62 there is some $g\colon V \to \mathbb{K}$ with $\|g\| = \|f\| = 1$ and $g|_{\mathbb{K}v} = f$. That means $\|g\| \leq 1 \Rightarrow \|e_V(v)\| \geq \|v\|$. And so we have:

77 **Lemma: Double Dual Map**

If $T\colon V_1 \to V_2$ is a bounded linear map then the following diagram commutes.

$$\begin{array}{ccc}
V_1 & \xrightarrow{T} & V_2 \\
\downarrow e_{V_1} & & \downarrow e_{V_2} \\
V_1'' & \xrightarrow{T''} & V_2''
\end{array}$$

Proof (77) All we have to do is apply the definitions of the dual objects for any $v \in V_1$ and $f \in V_2'$:

$$(e_{V_2} \circ T)(v)(f) = e_{V_2}(T(v))(f) = f(T(v))$$
$$(T'' \circ e_{V_1})(v)(f) = T''(e_{V_1}(v))(f) = e_{V_1}(v)(T'(f)) = T'(f)(v) = f(T(v))$$
$$\Rightarrow e_{V_2} \circ T = T'' \circ e_{V_1}$$

We say that (V, e_V, V'') is the natural *functor transformation* from $\mathcal{B}(V_1, V_2)$ into $\mathcal{B}(V_1'', V_2'')$ (not necessarily onto).

78 **Lemma: Adjoint of the Embedding**

For any vector space V and the natural embeddings $e_V : V \to V''$ and $e_{V'} : V' \to V'''$ we have $(e_V)' : V''' \to V'$ and:

$$(e_V)' \circ (e_{V'}) = \mathrm{id}_{V'}$$

Proof (78)

$$x \in V, f \in V' \Rightarrow f(x) = e_V(x)(f) = e_{V'}(f)(e_V(x)) = (e_{V'}(f) \circ e_V)(x)$$
$$= \left((e_V)'(e_{V'}(f))\right)(x) = \left((e_V)' \circ e_{V'}\right)(f)(x)$$
$$\Rightarrow f = \left((e_V)' \circ e_{V'}\right)(f)$$

79 **Definition: "Reflexive Normed Spaces"**

A normed space $(V, \|\cdot\|)$ is called *reflexive* iff $e_V : V \to V''$ is surjective.

Example

- For $V = c_0$ we get $c_0' = l_1$, $l_1' = l_\infty$ and l_∞' is the family of all bounded signed measures on $\mathbb{N}'$. So c_0 is not reflexive.
- Every finite-dimensional space is reflexive.

80 **Theorem: Properties of Reflexivity**

Let V be a normed vector space.

0. If V is reflexive, it is a Banach space.
1. Each closed linear subspace of V is reflexive if V is reflexive.
2. The quotient space V/L with L being a closed linear subspace, is reflexive if V is reflexive.
3. V is reflexive if there is a closed linear subspace L of V so that L and V/L are reflexive.
4. V' is reflexive iff V is reflexive.

Proof (80)

0. If V is reflexive, it is isometric to V'' and V'' is always Banach.

1.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \xrightarrow{\varepsilon_L} & V & \xrightarrow{\pi_L} & V/L & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle e_L} & & \downarrow{\scriptstyle e_V} & & \downarrow{\scriptstyle e_{V/L}} & & \\
0 & \longrightarrow & L'' & \xrightarrow{(\varepsilon_L)''} & V'' & \xrightarrow{(\pi_L)''} & (V/L)'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L''/L & \longrightarrow & 0 & \longrightarrow & (V/L)''/(V/L) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

   The columns and the first two rows are short exacts and so the third is too. So L''/L is trivial.

4. Because of $(e_V)' \circ (e_{V'}) = \mathrm{id}_{V'}$ we know that $e_{V'}$ is invertible if e_V and thus $(e_V)'$ is invertible.

   There is some $S \in \mathcal{B}(V'', V)$ with $S \circ e_V = \mathrm{id}_V$ so that

$$
\begin{aligned}
&(e_V)' \circ (S') = (S \circ e_V)' = (\mathrm{id}_V)' = \mathrm{id}_{V'} \\
&\left(\mathrm{id}_{V'}(f)\right)(v) = f(v) = f(\mathrm{id}_V(v)) = (\mathrm{id}_V)'(f)(v) \\
&(S') \circ (e_V)' = \mathrm{id}_{V'}
\end{aligned}
$$

In general it is not easy to see wheather some vector space is reflexive or not. So we introduce a stronger condition.

81 **Definition: "Strictly Normed, Strictly Convex, Uniformly Convex"**

Let $(V, \|\cdot\|)$ be a normed vector space. Then we say that V is *strictly normed* iff $\|x + y\| = \|x\| + \|y\|$ implies that x and y are linearly dependend over $\mathbb{R}$. V is called *strictly convex* iff for any $x, y \in V$ and $\alpha \in (0, 1)$:

$$\|x\| = \|y\| = \|\alpha x + (1 - \alpha)y\| = 1 \Rightarrow x = y$$

These two conditions are equivalent.

If there exists a continuous function $f\colon [0, 2] \to [0, 2]$ that is increasing and $f(0) = 0$ such that

$$\forall x, y \in S_V := \{\ x \in V \mid \|x\| = 1\ \} : \|x - y\| \le f(2 - \|x + y\|)$$

we call V *uniformly convex*.

Examples of non-uniformly convex Banach spaces
c_0, l_1, l_∞, $L_1(\mathbb{R})$, $L_\infty(\mathbb{R})$, $C_b(\mathbb{R})$, $C([0,1])$.

82 **Remark: Convexity**

1. The space of all measurable functions over X, $L_p(X,\Omega,\mu)$, is uniformly convex with $f(t) = \sqrt[p]{2-t^p}$.

2. For $x, y \in S_V$ and $0 < \alpha < 1$:

$$2 - \|x+y\| \leq (\min(\alpha, 1-\alpha))^{-1} (1 - \|\alpha x + (1-\alpha)y\|)$$

   So we can easily show, that uniform convexity implies strict convexity:

$$\begin{aligned}
&\|\alpha x + (1-\alpha)y\| = \|x\| = \|y\| = 1, \alpha \in (0,1) \\
&\Rightarrow \|x-y\| \leq f(2 - \|x+y\|) \leq f\left(\min(\alpha, 1-\alpha)^{-1}(1 - \|\alpha x + (1-\alpha)y\|)\right) \\
&= f(0) = 0 \\
&\Rightarrow x = y
\end{aligned}$$

   For $\|x\|, \|y\| \leq 1$ we can show

$$\|x-y\| \leq 1 - \|x\| + 1 - \|y\| + f\left(\min(2, 4 - (\|x+y\| - \|x\| - \|y\|))\right)$$

   Is $x = 0$ or $y = 0$ it is obvious. Otherwise we look at $\lambda_1 := \|x\|^{-1}$ and $\lambda_2 := \|y\|^{-1}$ so that $1 - \|x\| = \|(1-\lambda_1)x\|$ and $1 - \|y\| = \|(1-\lambda_2)y\|$.

$$\begin{aligned}
&\|x-y\| \leq \|\lambda_1 x - \lambda_2 y\| + (1 - \|x\|) + (1 - \|y\|) \\
&\|\lambda_1 x + \lambda_2 y\| \geq \|x+y\| - (1 - \|x\|) - (1 - \|y\|)
\end{aligned}$$

3. Is $L \subset V$ a closed subspace of V and V is strictly convex, L is strictly convex too with the same function f.

83 **Lemma: Uniformly Convexity Conditions**

Let $(V, \|\cdot\|)$ be a normed vector space. Then the following properties are equivalent:

1. V is uniformly convex

2. For any sequences $(x_n), (y_n)$ in S_V with $\lim_{n\to\infty} \|x_n + y_n\| = 2$ we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0$$

3. For any sequences $(x_n), (y_n)$ in S_V, $\|x_n\| \leq 1$, $\|y_n\| \leq 1$ and $\lim_{n\to\infty} \|x_n + y_n\| \geq 2$ implies $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Proof (83) (2) is a special case of (3). (3) follows easily from (1) with the triangle inequality. For the implication from (2) to (1) choose for $t \in [0,2]$:

$$\psi(t) := \sup \{ \|x - y\| \mid x, y \in V, 2 - t \leq \|x + y\|, \|x\| = 1, \|y\| = 1 \} \in [0,2]$$

Then ψ is increasing and

$$\inf \{ \psi(t) \mid t \in [0,2] \} = \lim_{t \downarrow 0} \psi(t)$$

We can find $x_n, y_n \in S_V$ so that

$$2 - \frac{1}{n} \leq \|x_n + y_n\| \wedge \psi(\frac{1}{n}) \leq \frac{1}{n} + \|x_n + y_n\| \leq \frac{1}{n} + \psi(\frac{1}{n})$$

which gives us $\lim_{n\to\infty} \|x_n + y_n\| = 2$ so according to (2) $\lim_{n\to\infty} x_n - y_n = 0$. But as ψ is not continuous yet, we have to choose a majorant function. For example I could choose the piecewise affine function with $f(2) = f(1) = 2$, $f(0) := 0$ and $f(\frac{1}{n+1}) := \psi(\frac{1}{n})$ for $n = 1, 2, \ldots$.

84 **Remark: Another Condition**

Another equivalent property is:

$$\limsup_{n\to\infty} \|x_n\| = \limsup_{n\to\infty} \|y_n\| = 1 \wedge \lim_{n\to\infty} \|x_n + y_n\| \geq 2 \Rightarrow \lim_{n\to\infty} \|x_n - y_n\| = 0$$

85 **Lemma: Cauchy Sequences in Uniformly Convex Spaces**

Let V be uniformly convex, then any sequence (x_n) in V with

$$\forall m, n \in \mathbb{N}: \|x_n + x_m\| \geq 2 \wedge \lim_{n\to\infty} \|x_n\| = 1$$

is a Cauchy sequence.

Proof (85)

$$\|x_m\| + \|x_n\| \geq \|x_n + x_m\| \geq 2$$
$$\Rightarrow \lim_{k\to\infty} \sup_{n,m\geq k} \|x_k + x_m\| \geq 2$$
$$\Rightarrow \lim_{n\to\infty} \|x_n - x_{n+k}\| = 0$$

86 **Lemma: Distance in Uniformly Convex Spaces**

Let $(V, \|\cdot\|)$ be a uniformly convex Banach space and $M \subset V$ a closed convex set. Then for every $v \in V$ there is exactly one $m \in M$ with $d(v, M) = \|v - m\|$.

8 The Milman Theorem and Related Aspects

87 **Lemma: Existence of Special Vectors**

Let $(V, \|\cdot\|)$ be a normed Banach space then for $G \in V''$ and $f_1, \ldots, f_n \in V'$ $\lambda \in (0,1)$, there exists $x \in V$ with $\|x\| \leq \|G\|$ and $f_j(x) = \lambda G(f_j), j = 1, \ldots, n$.

In general we cannot find such a x with $\|x\| = \|G\|$.

Proof (87) We only consider the non-obvious case $G \neq 0$. Let $T\colon V \to \mathbb{K}^n$ the continuous linear map given by $T(v) := (f_1(v), \ldots, f_n(v))$. Then the kernel $L := T^{-1}(0)$ is a closed linear subspace of V and $f_1, \ldots, f_n \in L^\perp$. Clearly V/L is normed by the quotient norm and $\dim(V/L) \leq n$ because $[T]\colon V/L \to \mathbb{K}^n$ is injective. So we get metrically exact sequences:

$$0 \longrightarrow L \longrightarrow V \xrightarrow{\pi_L} V/L \longrightarrow 0$$

$$0 \longleftarrow L' \longleftarrow V' \xleftarrow[(\pi_L)']{} (V/L)' \longleftarrow 0$$

$$0 \longrightarrow L'' \longrightarrow V'' \xrightarrow{(\pi_L)''} (V/L)'' \longrightarrow 0$$

Because V/L has dimension 1 it is reflexive.

$$G \in V'' \colon (\pi_L)''(G)(\tilde{f}_j) = G(f_j)$$

with $\tilde{f}_j := I(f_j)$ for the natural isometric isomorphism from $L^\perp$ onto $(V/L)'$. In the same way we get $\tilde{f}_j(\pi_L(v))$ for $j = 1, \ldots, n$. I is induced by the canonical transformations:

$$\begin{array}{ccccccccc}
0 & \longleftarrow & L' & \xleftarrow[\text{restriction}]{} & V' & \xleftarrow[(\pi_L)']{} & (V/L)' & \longleftarrow & 0 \\
 & & \Big\downarrow S & & & & \Big\uparrow I & & \\
0 & \longleftarrow & V'/L' & \xleftarrow[\pi_{V^\perp}]{} & V' & \longleftarrow & L^\perp & \longleftarrow & 0
\end{array}$$

$e_{V/L}\colon V/L \to (V/L)''$ is an isometric isomorphism and

$$\|(\pi_L)''\| = \|(\pi_L)'\| = \|\pi_L\| = 1 \Rightarrow \|(\pi_L)''(G)\| \leq \|G\|$$

π_L maps $U_V(0, \|G\|)$ onto $U_{V/L}(0, \|G\|)$ and $e_{V/L}^{-1} \circ (\pi_L)''(\lambda G) \in U_{V/L}(0, \|G\|)$, so there is some $x \in V$ with $x \in U_V(0, G)$ and $\pi_L(x) = e_{V/L}^{-1} \circ (\pi_L)''(\lambda G)$.

88 **Remark: Properties of Uniformly Convex Spaces**

Suppose that $(V, \|\cdot\|)$ is a uniformly convex normed vector space, let $g_1, g_2, \ldots$ be a sequence in V' with $\|g_j\| = 1$ and $(v_n)_{n \in \mathbb{N}}, v_n \in V$ a sequence with $\|v_n\| \leq 1$ and $g_j(v_n) \geq 1 - \frac{1}{j}$ for $j = 1, \ldots, n$.

1. Then (v_n) is a Cauchy sequence.
2. If $\lim_{n\to\infty} v_n = v$ exists in V, then $\|v\| = 1$.
3. If $x, y \in V$ satisfies $\|x\| = \|y\| = 1$ and $\lim_{j\to\infty} g_j(x) = \lim_{j\to\infty} g_j(y) = 1$ then $x = y$.

Proof (88)

1.

$$
\begin{aligned}
&2 \geq \|v_n + v_m\| \geq |g_j(v_n) + g_j(v_m)| \geq 2 - \frac{2}{j}, \quad j \leq \min(n, m) \\
&\Rightarrow 2 \geq \|v_n + v_m\| \geq 2 - \frac{2}{\min(n, m)} \\
&\Rightarrow \|v_n - v_m\| \leq (1 - \|v_n\|) + (1 - \|v_m\|) + \varphi(\min(2, \frac{2}{\min(n, m)})) \\
&\|v_n\| \geq |g_n(v_n)| \geq 1 - \frac{1}{n} \\
&\Rightarrow \|v_n - v_m\| \leq \frac{1}{n} + \frac{1}{m} + \varphi(\frac{2}{\min(n, m)})
\end{aligned}
$$

So (v_n) is Cauchy.

2.

$$
\|v\| = \lim_{n\to\infty} \|v_n\| = 1
$$

3. Let $\lim_{n\to\infty} g_n(x) = 1 = \lim_{n\to\infty} g_n(y)$ then

$$
\begin{aligned}
&\|x + y\| \geq |g_n(x) + g_n(y)| \xrightarrow{n\to\infty} 2 \\
&\Rightarrow \|x - y\| \leq \varphi(2 - \|x + y\|) = \varphi(0) = 0 \\
&\Rightarrow \|x + y\| = 2
\end{aligned}
$$

89 **Theorem: Milman**

Each uniformly convex Banach space is reflexive.

Proof (89) Let $G \in V''$, then we have to show that there exists $x \in V$ with $e_V(x) = G$, i.e. $\forall f \in V'\colon f(x) = G(f)$. For $f = 0$ this is clear and without loss of generality we can assume $\|G\| = 1$. According to the definition of the operator norm in $\mathcal{B}(V', \mathbb{K})$ there exists $g_n \in V'$ with $\|g_n\| = 1$ and $1 \geq |G(g_n)| \geq 1 - \frac{1}{n} \Rightarrow 1 \geq G(g_n) \geq 1 - \frac{1}{n}$ for each $n \in \mathbb{N}$.

Now let $f \in V'$ be arbitrary. By Lemma 87 there are $x_1, x_2, \ldots \in V$ with

$$
\begin{aligned}
&\|x_n\| \leq 1 \wedge g_j(x_n) = (1 - \frac{1}{n})G(g_j), \quad j = 1, \ldots, n \\
&f(x_n) = (1 - \frac{1}{n})G(f)
\end{aligned}
$$

And we get

$$
\begin{aligned}
&\Rightarrow \forall j = 1, \ldots, n\colon 1 \geq g_j(x_n) \geq (1 - \frac{1}{n})(1 - \frac{1}{j}) \\
&\Rightarrow 1 \geq g_j(x_n) \geq (1 - \frac{1}{n})^2 \geq 1 - \frac{2}{n}
\end{aligned}
$$

So the condition of Properties of Uniformly Convex Spaces applies and (x_n) is Cauchy.

As $(V, \|\cdot\|)$ is Banach the limit $x_f := \lim_{j\to\infty} x_j$ exists in V and

$$f(x_f) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} (1 - \frac{1}{n})G(f) = G(f)$$

We also get $\|x_f\| = 1$ because of $\|x_n\| \to 1$.

The same procedure can be done with any other $\tilde{f} \in V'$ and we would get $x_{\tilde{f}}$ with $\|x_{\tilde{f}}\| = 1$ and $\tilde{f}(x_{\tilde{f}}) = G(\tilde{f})$. Because for all $j \in \mathbb{N}$ we have $g_j(x_f) = G(g_j)$ and $g_j(x_{\tilde{f}}) = G(g_j)$ we can use Properties of Uniformly Convex Spaces again and get $x_f = x_{\tilde{f}}$ and so

$$\exists x \in V \forall g \in V' : g(x) = G(g)$$
$$\Rightarrow e_V(x) = G$$

90 **Lemma: Separability of Dual Spaces**

If V' is separable (in Norm) then V is separable.

Proof (90) Let $f_1, \dots, f_n, \dots$ be sequence in V' that is dense in Norm in V'. We can suppose that $\forall n: f_n \neq 0$. Find $x_n \in V$ with $f_n(x_n) \geq (1 - \frac{1}{2n})\|f_n\|$ and $\|x_n\| = 1$. Let $Y = \mathbb{Q}$ if $\mathbb{K}$ is real and $Y = \mathbb{Q} + i\mathbb{Q}$ if $\mathbb{K} = \mathbb{C}$. Let M be the Y-Module generated by $\{x_1, x_2, \dots\}$. M is countable and $\bar{M}$ is the closed linear subspace of V generated by $\{x_1, x_2, \dots\}$. We show that $\bar{M} = V$, otherwise $(V/\bar{M})' = \bar{M}^\perp = M^\perp$ is a non-zero linear subspace of V'. This is not possible because $\sup_n |f(x_n)| = \|f\|$ for any $f \in V'$.

9 Weak and *Weak Topologies

Let V and W be $\mathbb{K}$-vektor spaces and $\beta: V \times W \to \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ bilinear. Consider all $\mathbb{K}$-valued linear maps $\beta_w: V \to \mathbb{K}$ with $w \in W$. Then we write $\beta_w(v) := \beta(v, w)$.

91 **Definition: "σ-Topologies"**

We denote the coarsest (in other words weakest) topology on V so that all the maps β_w are continuous by $\sigma(V, W)$ and call it the $\sigma(V, W)$-topology.

Then of course any set $U \in \sigma(V, W)$ is a union of sets of the form: $\bigcap_{i=1}^n \beta_{w_i}^{-1}(U_i)$ with U_i open in $\mathbb{K}$ and $w_i \in W$. This means in particular, that the system of open neighbourhoods for $v_0 \in V$ with respect to the $\sigma(V, W)$-topology is given by

$$U(v_0, w_1, \dots, w_n, \varepsilon) := \left\{ v \in V \;\middle|\; \forall j = 1, \dots, n: \underbrace{|\beta(v, w_j) - \beta(v_0, w_j)|}_{\beta(v - v_0, w_j)} < \varepsilon \right\}$$

Conclusion If $\{ \beta(\cdot, w) \mid w \in W \}$ is separating for V, then the $\sigma(V, W)$-topology is Hausdorff, because the $\sigma(V, W)$-continuous functions separate the points of V.

Convergence in $\sigma(V, W)$ is then given by:

$$\lim_{n\to\infty,\sigma(V,W)} x_n = y \Leftrightarrow \forall w \in W\colon \lim_{n\to\infty} \beta(x_n, w) = \beta(y, w)$$

Now we consider the special case, that $(V, \|\cdot\|)$ and $(W, \|\cdot\|)$ are normed and that β is jointly continuous ($\sup_{\|v\|\le 1, \|w\|\le 1} |\beta(v, w)| =: \|\beta\| < \infty$. Then we have that the norm topology on V is finer than the $\sigma(V, W)$-topology. (i.e. $\mathrm{id}_V\colon (V, \|\cdot\|) \to (V, \sigma(V, W))$ is continuous) So in particular

$$\lim_{n\to\infty} \|x_n - y\| = 0 \Rightarrow y = \lim_{n\to\infty,\sigma(V,W)} x_n$$

Special Cases

- For $W := V'$ and $\beta(v, f) := f(v)$ the $\sigma(V, V')$-toplogy is called the *weak topology* on V.
- For $V := W'$ and $\beta(f, v) := f(w)$ the $\sigma(W', W)$-topology is called the **weak topology* on W'.

92 **Remark: Properties of (*)Weak Topology**

1. Each weakly continuous (linear) functional $f\colon V \to \mathbb{K}$ is in V', because $\mathrm{id}\colon (V, \|\cdot\|) \to (V, \text{weak})$ is continuous.
2. Convergence takes the special form:

$$\lim_{n\to\infty,\sigma(V,V')} x_n = y \Leftrightarrow \forall f \in V'\colon \lim_{n\to\infty} f(x_n) = f(y)$$

3. Each *weakly continuous linear functional $G\colon V' \to \mathbb{K}$ is contained in $e_V(V) \subset V''$

93 **Theorem: *Weakly Dense Embedding**

If V is Banach, $e_V(K_V(0, 1))$ is $\sigma(V'', V')$-dense (*weakly in V'') in $K_{V''}(0, 1)$.

Proof (93) Let $G_0 \in V''$ with $\|G_0\| \le 1$. Suppose that G_0 is not in the *weak closure of $e_V(K_V(0, 1))$. Then there exist $f_1, \dots, f_n \in V'$, $\varepsilon > 0$ such that

$$\begin{aligned} &\left\{ G \in V'' \;\middle|\; \forall j = 1, \dots, n\colon |G(f_j) - G_0(f_j)| < \varepsilon \right\} \cap e_V(K_V(0, 1)) \\ &= U(G_0, f_1, \dots, f_n, \varepsilon) \cap e_V(K_V(0, 1)) = \varnothing \end{aligned}$$

Let $\lambda \in (0, 1)$ such that $\forall j = 1, \dots, n\colon |\lambda G_0(f_j) - G_0(f_j)| < \varepsilon$. According to Lemma 87 there exists some $x \in V$ with $\|x\| \le \|G_0\| \le 1$ such that $e_V(x)(f_j) = f_j(x) = \lambda G_0(f_j)$ for $j = 1, \dots, n$. That means $G = e_V(x)$ is contained in the above intersection. Thus our assumption must be wrong.

94 **Theorem: Weakly Compactness**

$K_{V'}(0,1)$ is always *weakly compact. This is equivalent to the Axiom of Choice.

95 **Lemma: Metricability**

If V is a separable Banach space then $K_{V'}(0,1)$ is metricable for the $\sigma(V',V)$ topology.

Proof (95) Let (v_i) be a dense sequence in V (w.l.o.g. assume $v_n \neq 0$). Then we define

$$\rho(f,g) := \sum_{n=1}^{\infty} \|v_n\|^{-1} 2^{-n} |f(v_n) - g(v_n)| \leq \|f-g\| \sum_{n=1}^{\infty} 2^{-n} < \infty$$

Now we can check, that each functional $G = e_V(x)$ on V' is, restricted to $K_{V'}(0,1)$, continuous with respect to the metric topology, because $e_V(v_n)$ is continuous with Lipschitz constant $\|v_n\|2^n$. The function $e_V(v_n)\colon K_{V'}(0,1) \to \mathbb{K}$ are uniformly dense in the set of all functions $e_V(0) \in V''$. That means $\mathrm{id}\colon (K_{V'}(0,1),\rho) \to (K_{V'}(0,1),\sigma(V',V))$ is continuous. The other direction follows from the fact, that $(f,g) \mapsto |f(v_n) - g(v_n)|$ is continuous with respect to $\sigma(V',V)$.

96 **Corollary: Reflexivity Conditions**

Let $(V, \|\cdot\|)$ be Banach space. Then the following conditions are equivalent:

1. V is reflexive
2. $K_V(0,1)$ of $\|\cdot\|$ is weakly compact. (i.e. compact in $(V, \sigma(V,V'))$)
3. The *weak topology and the weak topology on V' coincide, i.e. $\sigma(V',V) = \sigma(V',V'')$.
4. $K_V(0,1)$ is sequentially compact.

Proof (96)

(4) ⇒ (2) follows from the Eberlein-Shmulian theorem

(1) ⇒ (2) We know that $e_V\colon V \to V''$ is an isometric isomorphism, so $e_V(K_V(0,1)) = K_{V''}(0,1)$ is $\sigma(V'',V')$-compact by Weakly Compactness. Since $f(v) = e_V(v)(f)$, we get that the topology $e_V^{-1}(\sigma(V'',V')) = \sigma(V,V')$. Since e_V is a bijective homeomorphism $K_V(0,1) = e_V^{-1}(K_{V''}(0,1))$ is compact too.

(2) ⇒ (1) $e_V\colon K_V(0,1) \hookrightarrow K_{V''}(0,1)$ is $\sigma(V,V')$-$\sigma(V'',V')$ continuous and $e_V(K_V(0,1))$ is dense in $K_{V''}(0,1)$ with respect to $\sigma(V'',V')$. The compactness of $K_V(0,1)$ implies that $e_V(K_V(0,1))$ is closed in $K_{V''}(0,1)$ with respect to $\sigma(V'',V')$. So density implies $e_V(K_V(0,1)) = K_{V''}(0,1)$ so that $e_V(V) = V''$.

(1) ⇒ (3) $e_V(V) = V''$ then for each $G \in V''$ there is a $v \in V$ with $e_V(v) = G$. Also $\sigma(V',V'')$ is the weakest topology on V' such that all the functions $G \in V''$ are continuous and $\sigma(V',V)$ is the weakest topology such that all the functions $e_V(v)$ for $v \in V$ are continuous. But as e_V maps V bijectively on V'' these conditions are equivalent and so the topologies coincide.

(3) ⇒ (1) Suppose that there is a $G \in e_V(V)$ with $\|G\|_{V''} = 1$. The function $G\colon V' \to \mathbb{K}$ is $\sigma(V', V'')$-continuous which is the same as $\sigma(V', V)$-continuous according to the precondition. So G is $\sigma(V', V)$-continuous on the compact space $K'_V(0,1)$. With the Stone-Weierstraß theorem we can approximate G uniformly by polynomials in functions $e_V(v_1), \dots, e_V(v_n)$ so that

$$\forall \varepsilon > 0 \exists v_1, \dots, v_n \in V, P(X_1, \dots, X_n)$$
$$\forall f \in K_{V''}(0,1)\colon |G(f) - P(f(v_1), \dots, f(v_n))| < \varepsilon$$
$$\operatorname{span}(v_1, \dots, v_n) \subset V, f \in \operatorname{Ker}(G) \Rightarrow |P(f(v_1), \dots, f(v_1))| < \varepsilon \|f\|$$
$$\Rightarrow \bigcap_{j=1}^{n} \operatorname{Ker}(e_V(v_j)) \subset \operatorname{Ker}(G)$$

Now let $\pi\colon f \in V \mapsto (f(v_1), \dots, f(v_n)) \in \mathbb{K}^n$. Then there is a linear $\lambda\colon \mathbb{K}^n \to \mathbb{K}$ so that $G = \pi \circ \lambda$ and so $G \in \operatorname{span}(e_V(v_1), \dots, e_V(v_n)) \Rightarrow G \in e_V(V)$ and so V is reflexive.

97 **Theorem: Tychonoff**

Let X be any nonempty set and $(Y_x)_{x \in X}$ a family of compact Hausdorff spaces then the cartesian product of these spaces $\mathcal{X}$ is compact with the Tychonoff topology $\mathcal{T}$. Where $\mathcal{T}$ is the weakest topology so that all the projections $\pi_x\colon \mathcal{X} \to Y_x, x \in X$ are continuous.

Is X countable, then there exists a proof in ZF otherwise the theorem is equivalent to the Axiom of Choice.

The Tychonoff topology can be described by a topological base. For $X_0 = \{x_1, \dots, x_n\} \subset X$ and open subsets $U_i \subset Y_{x_i}, i = 1, \dots, n$ we have:

$$U(x_1, \dots, x_n, U_1, \dots, U_n) := \left\{ f \in \mathcal{X} \;\middle|\; \forall j = 1, \dots, n\colon \pi_{x_j}(f) \in U_j \right\}$$

98 **Theorem: Special Case of the Tychonoff Theorem**

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $Y_x = Y := \{ \alpha \in \mathbb{K} \mid |\alpha| \le 1 \}$. Then the product is the *Hilbert cube* $\mathcal{X} = Y^X = \{f\colon X \to Y\}$ and the Tychonoff topology on $\mathcal{X}$ is the topology of pointwise convergence.

If $X = \mathbb{N}$, then $\mathcal{X} = Y^{\mathbb{N}}$ is metrizable by

$$\rho(f, g) := \frac{1}{2} \left(\sum_{n=1}^{\infty} 2^{-n} |f(n) - g(n)| \right)$$

I.e. $\mathcal{O}_\rho$ is the Tychonoff topology on $Y^{\mathbb{N}}$. Then the compactness can be proven with Cantors diagonalization.

99 **Lemma: Series Spaces**

Let $X \neq \emptyset$ be a set and $V := \ell_1(X, \mathbb{K}) := \left\{ f\colon X \to \mathbb{K} \;\middle|\; \sum_{x \in X} |f(x)| < \infty \right\}$. Then V is a Banach space with $\|f\|_1 = \sum_{x \in X} |f(x)|$. The space $(K_{V'}(0,1), \sigma(V', V))$ is naturally isomorphic to Y^X with $Y = K_{\mathbb{K}}(0,1)$ and the topology of pointwise convergence.

Proof (94) (idea)

1. Let $X \subset S_{\|\cdot\|}(1) \subset V$ be a dense subset of $K_{V'}(0,1)$ and define $T\colon \ell_1(X,\mathbb{K}) \to V$ by $T(f) := \sum_{x\in X} f(x)\cdot x$, which is absolutely convergent. Then $\|T\| \le 1$ and $T(K_{\ell_1(X,\mathbb{K})}(0,1))$ is dense in $K_V(0,1)$. This implies for $L := \operatorname{Ker} T$ the map $[T]\colon \ell_1(X,\mathbb{K})/L \to V$ is an isometric isomorphism. Notice $\ell_\infty(X,\mathbb{K}) \simeq \ell_1(X,\mathbb{K})$.

2. Thus $T'(K_{V'}(0,1)) = \left(K_{\ell_\infty(X,\mathbb{K})}(0,1)\right) \cap L$ which is a closed subset of the (according to Tychonoff) compact space $K_{\ell_1(X,\mathbb{K})'}(0,1)$ with the *weak-topology.

3. T' is a homeomorphism from $K_{V'}(0,1)$ onto the closed subset $K_{\ell_\infty(X,\mathbb{K})}(0,1) \cap L^\perp$ of $\ell_\infty(X,\mathbb{K})$ with the $\sigma(\ell_\infty(X,\mathbb{K}),\ell_1(X,\mathbb{K}))$-topology. And so $K_{V'}(0,1)$ is compact.

100 **Remark: Strict Norms**

1. $(V,\|\cdot\|)$ ist strictly normed iff it is strictly convex.

2. If $(V,\|\cdot\|)$ is uniformly convex, it is strictly convex.

101 **Theorem: Convergence Conditions**

Let $(V,\|\cdot\|)$ uniformly convex Banach space. Then

1. If $x_n \to y$ weakly (i.e. $f(x_n) \to f(x)$ for any $f \in V'$) and $\|x_n\| \to \|y\|$, then $\|x_n - y\| \to 0$.

2. If $x \in V$ and M is a closed convex subset of V then there is a unique element $y \in M$ such that $\|x - y\| = \min_{m\in M} \|x - m\|$.

Proof (101)

1. W.l.o.g. we assume $\|y\| = 1$ and $\|x_n\| = 1$. By Theorem 62 (Hahn-Banach Extension Theorem) there is some $f \in V'$ with $f(y) = 1, \|f\| = 1$ so that

$$2 \ge \|x_n + y\| \ge |f(y) + f(x_n)| \to 2 \Rightarrow \|x_n - y\| \le \varphi(2 - \|x_n + y\|) \to 0$$

2. Let $\rho := \inf_{m\in M} \|x - m\|$. If $\rho = 0$ then $x \in M = \bar{M}$. Suppose $\rho > 0$. Then replace M by $\rho^{-1}(M - x)$ and x by 0. Now we have to show that there is a unique $y \in M$ with $\|y\| = 1$.

   Existence There is a sequence $(x_n), x_n \in M$ with $1 \le \|x_n\| \to 1$. By convexity of M

$$\forall m,n \in \mathbb{N}\colon \|x_n + x_m\| = 2\|\underbrace{\tfrac{1}{2}x_n + \tfrac{1}{2}x_m}_{\in M}\| \ge 2$$
$$\Rightarrow \inf_{m,n\ge k} \|\|x_m\|^{-1}x_m + \|x_n\|^{-1}x_n\| \to 2, k \to \infty$$
$$\Rightarrow \|\|x_m\|^{-1}x_m - \|x_n\|^{-1}x_n\| \le \varphi(2 - \|\|x_m\|^{-1}x_m - \|x_n\|^{-1}x_n\|)$$
$$\to 0, \min(n,m) \to 0$$

   Thus (x_n) is Cauchy and because V is Banach and M is closed we get:

$$y = \lim_{n\to\infty} x_n \in M \wedge \|y\| = \lim_{n\to\infty} \|x_n\| = 1$$

Uniqueness Let $y' \in M$ with $\|y'\| = 1$, $1 \geq \|\frac{1}{2}y + \frac{1}{2}y'\| \geq 1$ and so $y' = y$ because of strict convexity.

According to the 62: If V is a Banach space and $X \subset V$, $S \subset V'$ we define

$$X^\circ := \left\{ f \in V' \;\middle|\; \forall x \in X\colon \operatorname{Re} f(x) \leq 1 \right\}$$
$$S_\circ := \left\{ v \in V \;\middle|\; \forall f \in S'\colon \operatorname{Re} f(v) \leq 1 \right\}$$

then $(X^\circ)_\circ$ is the norm-closure of the convex hull of $X \cup \{0\}$.

X° is always a $\sigma(v', V)$-closed convex subset of V' containing 0 and $S_\circ$ is always a $\sigma(V, V')$-closed subset of V.

10 Spectra of Bounded Linear Maps

102 **Definition: "Spectrum, Resolvent Set, Eigenvalue"**

Let V be a complex Banach space $T \in \mathcal{B}(V) := \mathcal{B}(V, V)$ then we define the *spectrum* of T as:

$$\operatorname{spec} T := \left\{ \lambda \in \mathbb{C} \;\middle|\; T - \lambda \operatorname{id}_V \text{ is not invertible in } \mathcal{B}(V) \right\}$$

and the complement is the *resolvent set*:

$$\operatorname{res} T := \mathbb{C} \setminus \operatorname{spec} T = \left\{ \lambda \in \mathbb{C} \;\middle|\; (T - \lambda \operatorname{id}_V)^{-1} \text{ exists} \right\}$$

$\lambda \in \mathbb{C}$ is called an *eigenvalue* of T if there is a $v \in V$ so that $(T - \lambda)(v) = 0$. So then all the eigenvalues are contained in the spectrum.

103 **Theorem: Properties of the Spectrum**

Let V be a Banach space over $\mathbb{C}$.

0. Every eigenvalue of T is in $\operatorname{spec} T$ and for $\dim V < \infty$ every element of $\operatorname{spec} T$ is an eigenvalue.
1. The set of invertible elements of $\mathcal{B}(V)$ is open in $\mathcal{B}(V)$ and $\operatorname{res} T$ is an open subset of $\mathbb{C}$.
2. The *spectralradius* satisfies:
$$\mathrm{r}(T) := \limsup_{n\to\infty} \|T^n\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \|T^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = \lim_{k\to\infty} \|T^{2^k}\|^{2^{-k}}$$
3. $\{ z \in \mathbb{C} \mid |z| > r(T) \} \subset \operatorname{res} T$
4. $\operatorname{spec} T$ is a closed subset of $\mathbb{C}$ and $\mathrm{r}(T) = \sup \{ |z| \mid z \in \operatorname{spec} T \}$
5. $R(T)\colon \operatorname{res} T \to \mathcal{B}(V), z \mapsto (T - z)^{-1} =: R_z(T)$ is a complex analytic function in the following sense:

   For every $z_0 \in \operatorname{res}(T)$ there are some elements $T_0, T_1, \ldots \in \mathcal{B}(V)$ such that $R_z(T) = \sum_{n=0}^{\infty} (z - z_0)^n T_n$ where we have absolute convergence for $|z - z_0| < \|(T - z_0)^{-1}\|^{-1} = \|R_{z_0}(T)\|^{-1}$.

6. $\operatorname{spec} T \neq \emptyset$ for any $T \in \mathcal{B}(V)$ if $\dim V > 0$.
7. For $z_0, \alpha \in \mathbb{C}$ we have the following rules
   a) $\operatorname{spec}(T + z_0 \operatorname{id}_V) = \operatorname{spec}(T) + z_0$
   b) $\operatorname{spec}(\alpha T) = \alpha \operatorname{spec} T$
   c) $\operatorname{spec} T' = \operatorname{spec} T$

104 **Lemma: Inversion Criteria**

Let V, W be Banach spaces and $T_0 \colon V \to W$ bijective as well as continuous. Then each $T \in \mathcal{B}(V, W)$ with $\|T_0 - T\| < \|T_0^{-1}\|^{-1}$ is invertible and

$$T^{-1} = \sum_{n=0}^{\infty} \left(T_0^{-1}(T_0 - T)\right)^n \circ T_0^{-1}$$

In particular if $T \in \mathcal{B}(V)$, $z_0 \in \operatorname{res} T$ and $|z - z_0| = \|(T - z_0) - (T - z)\| < \|R_{z_0}(T)\|^{-1}$ then $(T - z)^{-1} = \sum_{n=0}^{\infty} (z - z_0)^n (R_{z_0}(T))^{n+1}$.

Proof (104)

$$T = T_0(1 - T_0^{-1}(T_0 - T))$$
$$\|T_0 - T\| \leq \|T_0^{-1}\|^{-1} \Rightarrow \|T_0^{-1}(T_0 - T)\| < 1$$

Let $S = T_0^{-1}(T_0 - T)$. Then $\|S\| < 1$ and $\sum_{n=0}^{\infty} S^n$ is absolutely convergent:

$$\sum_{n=0}^{\infty} \|S^n\| \leq \sum_{n=0}^{\infty} \|S\|^n = \frac{1}{1 - \|S\|}$$

As $\mathcal{B}(V)$ is Banach there is some $L \in \mathcal{B}(V)$ with $L = \sum_{n=0}^{\infty} S^n$ and so $L(1 - S) = 1 = (1 - S)L$. So then $T = T_0(1 - S) \Rightarrow T^{-1} = (1 - S)^{-1} T_0^{-1} = \sum_{n=0}^{\infty} (T_0^{-1}(T_0 - T))^n T_0^{-1}$.

Proof (103)

1. Is implied by the above lemma.
5. Holds by the above Lemma for $T_n := \left(R_{z_0}(T)\right)^{n+1}$.
3. Let $|z| > r(T) = \limsup_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ then

$$T - z = -z(1 - z^{-1}T)$$
$$\Rightarrow (T - z)^{-1} = -z^{-1}(1 - z^{-1}T)^{-1} = -z^{-1} \sum_{n=0}^{\infty} \left(z^{-1}T\right)^n$$
$$\|(z^{-1}T)^n\| = |z|^{-n} \|T^n\|$$

$\sum_{n=0}^{\infty} (z^{-1}T)^n$ is absolutely convergent if

$$\limsup_{n\to\infty} \sqrt[n]{\|(z^{-1}T)^n\|} = |z^{-1}| \limsup_{n\to\infty} \|T^n\|^{\frac{1}{n}} < 1$$

(This is the *Euler criterion*)

We also see that $\operatorname{spec} T \subset r(T)D_2$ and so $\operatorname{spec} T$ is compact.

105 **Lemma: Generalized Liouville Theorem**

Let V be a Banach space, $U \subset \mathbb{C}$ open and $F\colon U \to \mathcal{B}(V)$ a analytic map. Then

1. For all $v \in V, g \in V'$ the map $z \in U \mapsto g(F(z)v) \in \mathbb{C}$ is analytic on U.
2. If $U = \mathbb{C}$ and F is bounded, $F(z) = F(0)$ for all $z \in \mathbb{C}$.
3. If $g(F(z)v) = \sum_{n=0}^{\infty} g((z^n T_n)(v)) = \sum_{n=0}^{\infty} z^n g(T_n v_n)$ for some $\varepsilon > 0$ and all $|z| < \varepsilon$ and F is defined on $\{\, z \mid |z| \le \alpha \,\}$ for some $\alpha \ge \varepsilon$. Then $\limsup_{n\to\infty} \|T_n\|^{\frac{1}{n}} \le \frac{1}{\alpha}$.

Proof (105)

1. This is obvious by the definition of analytic function.
2. If $\sup_{z\in\mathbb{C}} \|F(z)\| = G$, we can conclude that $f(z) := g(F(z)v)$ is holomorphic on $\mathbb{C}$ and $\sup_{z\in\mathbb{C}} |f(z)| \le \|g\|\|v\|G < \infty$. So according to the classical Liouville Theorem f must be constant and thus $\forall z \in \mathbb{C}, g \in V', v \in V\colon g(F(z)v) = g(F(0)v) \Rightarrow \forall z \in \mathbb{C}\colon F(z) = F(0)$.
3. Let $f(z) := g(F(z)v)$ for $z \in U$. Since $F(z)$ is analytic on U and $\alpha \operatorname{Int} D_2 \subset U$ we have that $f(z)$ is holomorphic on $\alpha \operatorname{Int} D_2 = \{\, z \mid |z| < \alpha \,\}$. On the other hand $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ for $\alpha_n = g(T_n v_n)$ and all $|z| < \varepsilon \le \alpha$.

   This implies $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ for all $|z| < \alpha$ and thus $|z| \limsup_{n\to\infty} |\alpha_n|^{\frac{1}{n}} \le 1$ for all $z, |z| < \alpha$ and so $\limsup_{n\to\infty} |\alpha_n|^{\frac{1}{n}} \le \frac{1}{\alpha}$. Moreover we have that $|z^n g(T_n v)| \xrightarrow{n\to\infty} 0$ for all $z, |z| < \alpha$. That means $z^n T_n v \in V$ converges weakly to zero for all $|z| < \alpha$. Then according to Corollary 47 (Banach-Steinhaus Theorem, Uniform-Boundedness Principle):

   $$\forall z, |z| < \alpha \exists C(v,z) \in \mathbb{R}\colon \sup_{n\in\mathbb{N}} \|z^n T_n(v)\| \le C(v,z) < \infty$$

   applying Corollary 47 again we get:

   $$\forall z, |z| < \alpha \exists C(z) < \infty\colon \sup_{n\in\mathbb{N}} |z|^n \|T_n\| \le C(z) < \infty$$

   $$\Rightarrow \forall z, |z| < \alpha\colon \limsup_{n\to\infty} \|T_n\|^{\frac{1}{n}} \le 1$$

   $$\Rightarrow \limsup_{n\to\infty} \|T_n\|^{\frac{1}{n}} \le \frac{1}{\alpha}$$

Proof (103) continued

6. Let $|z| \geq \|T\| + 1$ then, because $T - z = (-z)(1 - z^{-1}T)$ we know that

$$R_z(T) = -z^{-1} \sum_{n=0}^{\infty} (z^{-1}T)^n$$
$$\Rightarrow \|R_z(T)\| \leq |z|^{-1} \sum_{n=0}^{\infty} \left(\frac{\|T\|}{|z|}\right)^n = |z|^{-1} \frac{1}{1 - \frac{\|T\|}{2}} = \frac{1}{|z| - \|T\|} \leq 1$$
$$\Rightarrow \forall z, |z| \geq \|T\| + 1 \colon \|R_z(T)\| \leq 1$$

Now assume that $\operatorname{spec} T = \emptyset \Leftrightarrow \operatorname{res} T = \mathbb{C}$, then $z \in (\|T\|+1)D_2 \mapsto R_z(T) \in \mathcal{B}(V)$ is continuous. Since $(\|T\|+1)D_2$ is compact and the function $z \mapsto \|R_z(T)\|$ is continuous we get that it assumes its maximum in some point $z_0 \in (\|T\|+1)D_2$, i.e. $\sup_{z,|z|\leq 1+\|T\|} \|R_z(T)\| \leq \|R_{z_0}(T)\|$. Thus $\sup_{z\in\mathbb{C}} \|R_z(T)\| \leq \max(1, \|R_{z_0}(T)\|) < \infty$.

So now Lemma 105 (Generalized Liouville Theorem) applies and gives that $R_z(T) = R_0(T)$ for all $z \in \mathbb{C}$. But $R_0(T) = T^{-1} = (T-1)^{-1} \Rightarrow T = T-1 \Rightarrow 0 = 1$ which is a contradiction. Thus we have shown, that $\operatorname{spec} T \neq \emptyset$.

4. $\sup \{ |z| \mid z \in \operatorname{spec} T \} = r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$. We know that $|z| > r(T) \Rightarrow z \in \operatorname{res} T$. Suppose that $r_1 \leq r(T)$ with $\operatorname{spec} T \subset r_1 D_2$. Then $\left\{ z \in \mathbb{C} \,\middle|\, |z| > r_1 \right\} \subset \operatorname{res} T$.

$$R_z(T) = -z^{-1}(1 - z^{-1}T)^{-1} = (T - z)^{-1}, \quad |z| > r_1$$
$$F(\lambda) := (1 - \lambda T)^{-1} = \lambda^{-1} R_{\lambda^{-1}}(T)$$

F is well-defined and holomorphic for $|\lambda| < \frac{1}{r_1}$. If $|\lambda| > r(T)$ then $(1-\lambda T)^{-1} = \sum_{n=0}^{\infty} \lambda^n T^n$. Since $F(\lambda)$ is holomorphic in $|\lambda| < \frac{1}{r_1}$ we get by Lemma 105 (Generalized Liouville Theorem) that $r(T) = \limsup_{n\to\infty} \|T^n\|^{\frac{1}{n}} \leq r_1$ and $F(\lambda) = \sum_{n=0}^{\infty} \lambda^n T^n$ for all $\lambda, |\lambda| < \frac{1}{r_1}$

7. a) $z \in \operatorname{res}(T + z_0) \Leftrightarrow (T + z_0 - z)^{-1}$ exists $\Leftrightarrow z - z_0 \in \operatorname{res}(T) \Rightarrow \operatorname{res}(T) + z_0 = \operatorname{res}(T + z_0) \Rightarrow \operatorname{spec}(T) + z_0 = \operatorname{spec}(T + z_0)$

   b) $\alpha = 0$ is ok, so now look at $\alpha \neq 0$:

$$z \in \operatorname{res}(\alpha T) \Leftrightarrow (\alpha T - z)^{-1} = \alpha^{-1}(T - \frac{z}{\alpha})^{-1} \text{ exists} \Leftrightarrow \frac{z}{\alpha} \in \operatorname{res} T$$
$$\Rightarrow \operatorname{res}(\alpha T) = \alpha \operatorname{res} T \Leftrightarrow \operatorname{spec}(\alpha T) = \alpha \operatorname{spec} T$$

   c) $T' - z = (T - z)'$, $(TS)' = S'T'$ and $(\mathrm{id}_V)' = \mathrm{id}_{V'}$.

106 **Theorem: Rational Spectrum Invariance**

Let $Q \colon \mathbb{C} \to \mathbb{C}$ be a rational function, $T \in \mathcal{B}(V)$, V a Banach space and suppose that the poles of Q are disjoint from $\operatorname{spec} T$. Then

$$\operatorname{spec}(Q(T)) = Q(\operatorname{spec}(T))$$

Proof (106) By assumption we can find $P_1, P_0 \in \mathbb{C}[X]$ so that $Q = P_1/P_0$ where P_1 and P_0 are relatively prime (i.e. do not have a common root). We can write $P_0(T)$ as $\alpha(T - z_1)(T - z_2)\ldots(T - z_n)$ and the poles of Q are those z_i, meaning $z_i \notin \operatorname{spec}(Q)$. That is why the $(T - z_i), i = 1, \ldots, n$ and thus P_0 are invertible in $\mathcal{B}(V)$. So $Q(T) := P_0(T)^{-1} P_1(T)$ exists in $\mathcal{B}(V)$.

Let $\omega \in \mathbb{C}$ and define $S_\omega(z) := P_1(z) - \omega P_0(z)$ then:

$$S_\omega(z) = 0 \Leftrightarrow P_0(z) \neq 0 \wedge Q(z) = \omega$$

Let $S_\omega(z) = \beta_\omega(z - z_1(\omega))(z - z_2(\omega))\ldots(z - z_{m_\omega}(\omega))$, $m_\omega = \operatorname{grad} S_\omega \leq \max(\operatorname{grad} P, \operatorname{grad} P_0)$ so that

$$P_0(z) \neq 0 \wedge Q(z) = \omega \Leftrightarrow z \in \{z_1(\omega), z_2(\omega), \ldots, z_m(\omega)\}$$
$$P_0(T)^{-1} S_\omega(T) = P_0(T)^{-1} P_1(T) - \omega = Q(T) - \omega$$

Thus

$$S_\omega(T) \text{ invertible } \Leftrightarrow \omega \in \operatorname{res} Q(T)$$
$$S_\omega(T) = \beta_\omega \prod_{i=1}^{m_\omega} (T - z_i) \text{ invertible } \Leftrightarrow \forall i = 1, \ldots, m_\omega : T - z_i(\omega) \text{ invertible}$$
$$\omega \in \operatorname{res} Q(T) \Leftrightarrow \forall i = 1, \ldots, m_\omega : z_i(\omega) \in \operatorname{res} T$$
$$\omega \in \operatorname{spec} Q(T) \Leftrightarrow \exists i_0 \in \{1, \ldots, m_\omega\} : z_i(\omega) \in \operatorname{spec} T$$
$$\exists i_0 : z_i(\omega) \in \operatorname{spec} T \Leftrightarrow \exists z \in \operatorname{spec} T : S_\omega(z) = 0 \Leftrightarrow \exists z \in \operatorname{spec} T : Q(z) = \omega$$
$$\omega \in \operatorname{spec} Q(T) \Leftrightarrow \exists z \in \operatorname{spec} T : Q(z) = \omega \Leftrightarrow \operatorname{spec} Q(T) = Q(\operatorname{spec} T)$$

11 Compact Operators

107 **Definition: "Compact Operators"**

Let V, W be Banach spaces, then the linear map $T: V \to W$ is called *compact*, if it satisfies one of the following, equivalent conditions:

1. The image $T(M)$ of each bounded subset $M \subset V$ has a compact closure in W.
2. The image $T(U_V(0, 1))$ of the open unit ball is relatively compact in W (meaning precompact, as W is Banach).
3. If $(v_n) \in V^{\mathbb{N}}$ is a bounded sequence, then $(T(v_n))_{n \in \mathbb{N}}$ contains a convergent subsequence $(T(v_{n_k}))_{k \in \mathbb{N}}$. Because W is Banach it suffices to show that $(T(v_{n_k}))$ is Cauchy.

108 **Definition: "Fredholm Operator"**

Let V, W be Banach spaces then $T \in \mathcal{B}(V, W)$ is called *Fredholm operator* iff

1. $\operatorname{Ker} T$ has finite dimension.

2. $W/T(V)$ has finite dimension.

These conditions notably imply that $T(V)$ is closed in W.

The (analytic) *index* of a Fredholm operator T is given by $\dim(\operatorname{Ker} T) - \dim(\operatorname{CoKer} T)$ with $\operatorname{CoKer} T := W/T(V)$.

Examples of Compact Operators

1. Let $f \in C([0,1])$ and $k \in C([0,1]^2)$. Then we get the compact integral operator:

$$T_k(f)(s) := \int_0^1 k(s,t) f(t) \mathrm{d}t$$

2. The same can be done for $f \in L_2((0,1))$ and $k \in L_2((0,1)^2)$.
3. Any operator with finit rank, i.e. $T \in \mathcal{B}(V,W)$ and $\dim(T(V)) < \infty$.

Examples of Fredholm Operators

1. If $T: V \to V$ is compact, $\mathrm{id}_V - T$ is Fredholm with $\operatorname{ind}(\mathrm{id}_V - T) = 0$.
2. The unilateral shift operator $S: \ell_2 \to \ell_2$, $S(a_1, a_2, a_3, \ldots) = (0, a_1, a_2, \ldots)$ is Fredholm with $\operatorname{ind} S = -1$.

109 **Theorem: Basics on Compact Operators**

Let U, V, W be Banach spaces and denote by $\mathcal{K}(V,W)$ the set of compact operators $T: V \to W$.

0. $\mathcal{K}(V,W) \subset \mathcal{B}(V,W)$
1. $\mathcal{K}(V,W)$ is a closed linear subspace of $\mathcal{B}(V,W)$.
2. ideal-property of $\mathcal{K}(V,W)$:
   a) $\mathcal{K}(V,W) \circ \mathcal{B}(U,V) \subset \mathcal{K}(U,W)$
   b) $\mathcal{B}(V,W) \circ \mathcal{K}(U,V) \subset \mathcal{K}(U,W)$
   c) If $V_1 \subset V$, $W_1 \subset W$ are closed subspaces and $T \in \mathcal{K}(V,W)$, $T(V_1) \subset W_1$ then $T|_{V_1} \in \mathcal{K}(V_1, W_1)$ and $[T]_{V_1,W_1}: V/V_1 \to W/W_1 \in \mathcal{K}(V/V_1, W/W_1)$.
   d) If $I: W \to U$ is isometric, $T \in \mathcal{B}(V,W)$ and $I \circ T \in \mathcal{K}(V,U)$ implies $T \in \mathcal{K}(V,W)$.
3. In particular $\mathcal{K}(V) := \mathcal{K}(V,V)$ is a closed ideal of $\mathcal{B}(V) = \mathcal{B}(V,V)$.
4. For any $T \in \mathcal{K}(V,W)$ the set $\overline{T(V)}$ is a separable, closed, linear subspace of W.
5. $T \in \mathcal{K}(V,W) \Leftrightarrow T' \in \mathcal{K}(W', V')$
6. (Riesz-Schauder Theorem) Let $T \in \mathcal{K}(V)$, then the operator $S := \mathrm{id}_V - T$ has the following properties:

a) $\dim(\operatorname{Ker} S) < \infty$

b) $S(V)$ is closed in V.

c) $\dim(V/S(V)) =: \operatorname{codim} S(V) < \infty$

d) $\dim(V/S(V))' = \dim(\operatorname{Ker}(S')) < \infty$ (moreover $(V/S(V))' \simeq \operatorname{Ker} S'$)

110 **Remark: Compact and Fredholm Operators**

The Riesz-Schauder-Theorem says in other words that $S = \mathrm{id}_V - T$ is a Fredholm operator for each $T \in \mathcal{K}(V)$. We shall see that $\dim(\operatorname{Ker} T) = \operatorname{codim}(T(V))$, i.e. $S = \mathrm{id}_V - T$ is a Fredholm operator of index zero.

111 **Theorem: Riesz**

Let V be a Banach space and $T \in \mathcal{K}(V)$.

1. If $\dim V = \infty$, then $0 \in \operatorname{spec} T$.

2. Each $\lambda \in \operatorname{spec} T \setminus \{0\}$ is an eigenvalue of T. The corresponding eigenspace $\operatorname{Ker}(T - \lambda \,\mathrm{id})$ is of finite dimension, the image $(T - \lambda \,\mathrm{id})(V)$ is closed in V and
$$\dim(\operatorname{Ker}(T - \lambda)) = \dim(V/(T - \lambda)(V)) =: \operatorname{codim}((T - \lambda)(V)) = \dim(\operatorname{Ker}(T' - \lambda))$$

3. $\operatorname{spec} T$ is at most countable and $(\operatorname{spec} T) \setminus \{0\}$ is discrete.

4. For each $\lambda \in (\operatorname{spec} T) \setminus \{0\}$ there exists a topological space $V = N(\lambda) \oplus R(\lambda)$ with $T(N(\lambda)) \subset N(\lambda), T(R(\lambda)) \subset R(\lambda)$ where $N(\lambda)$ is of finite dimension, $\operatorname{Ker}(T - \lambda) \subset N(\lambda)$ and $(T - \lambda)|_{R(\lambda)}$ is an homomorphism from $R(\lambda)$ onto $R(\lambda)$.

112 **Corollary: Fredholm Alternative**

Let $T \in \mathcal{K}(V)$, V a Banach space and $\lambda \in \mathbb{C} \setminus \{0\}$. Then one of the following alternatives occur:

1. The homogeneous equation $(\lambda - T)v = 0$ has no other solution than $v = 0$. In this case $\lambda - T$ is invertible, i.e. the inhomogeneous equation $(\lambda - T)v = x, x \in V$ has exactly one solution for each x.

2. There exist n linear indepentent solutions $v_1, \ldots, v_n \in V$ of the equation $(\lambda - T)v = 0$. In this case the equation $(\lambda - T')v' = 0$ has also n linear indepentent solutions and the inhomogeneous equation $\lambda v - Tv = x, x \in V$ has a solution iff $x \in (\operatorname{Ker}(\lambda - T'))_\perp = \overline{(\lambda - T)(V)} = (\lambda - T)(V)$.

113 **Remark: Fredholm Operators Space**

We can show that

1. The set $\mathcal{F}(V, W)$ of Fredholm operators is open in the Banach space $\mathcal{B}(V, W)$.

2. $\mathrm{ind}\colon S \in \mathcal{F}(V, W) \mapsto \dim(\operatorname{Ker}(S)) - \operatorname{codim}(S(V)) \in \mathbb{Z}$ is a continuous map.

This can be applied to show that $\mathrm{ind}(\mathrm{id} - T) = 0$ for $T \in \mathcal{K}(V)$, because $\lambda \in [0, 1] \mapsto \lambda T \in \mathcal{K}(V)$ is a norm-continuous linear map and thus $\mathrm{ind}(\mathrm{id}_V - T) = \mathrm{ind}(\mathrm{id}_V - 0) = 0$.

114 **Lemma: Subspace Sequences**

Let V be a Banach space, $T \in \mathcal{B}(V)$, $W_1 \subset W_2 \subset \ldots \subset V$ a sequence of closed linear subspace of V and a sequence $(\lambda_n)_{n\in\mathbb{N}}$, $\lambda_n \in \mathbb{K} \setminus \{0\}$ such that $(\lambda_n \operatorname{id} - T)W_n \subset W_{n-1}$ for $n = 2, 3, \ldots$.

1. Then $\forall n \in \mathbb{N}\colon T(W_n) \subset W_n$ and if $W_{n_0} \neq W_{n_0+1}$ there exists $x_{n_0} \in W_{n_0}$ with $\|x_{n_0}\| \leq 2$ and

$$\|\pi_{W_{n_0-1}}(Tx_{n_0})\| = \operatorname{dist}(Tx_{n_0}, W_{n_0-1}) \geq |\lambda_{n_0}|$$

2. In particular if there exist $n_1 < n_2 < \ldots$ such that $W_{n_k} \neq W_{n_k-1}$ then there are $x_{n_k} \in W_{n_k}$ such that $\|x_{n_k}\| \leq 2$ and $\|Tx_{n_k} - Tx_{n_l}\| \geq |\lambda_{n_k}|$ for all $k > l$.

3. If T is compact, then one of the following properties hold:

$$\exists p \in \mathbb{N} \forall n \in \mathbb{N}\colon W_p = W_{p+n}$$
$$\inf_{n\in\mathbb{N}} |\lambda_n| = 0$$

Proof (111)

1. If $0 \notin \operatorname{spec} T$, T^{-1} exists and so $\operatorname{id}_V = T^{-1}T$ is compact. Thus $U(0,1)$ is precompact in V and V must be finit-dimensional.

2. follows from Riesz-Schauder Theorem.

3. Let $\lambda_1, \lambda_2, \ldots \in \operatorname{spec}(T) \setminus \{0\}$ be a sequence, $n \neq m \Rightarrow \lambda_n \neq \lambda_m$ and $|\lambda_1| \geq |\lambda_2| \geq \ldots$ then $\lim_{n\to\infty} \lambda_n = 0$ would imply that $\operatorname{spec}(T) \setminus \{0\}$ is discrete. As eigenvectors x_n of T for different eigenvalues λ_n are linearly independent we can define $W_n := \operatorname{span}(x_1, \ldots, x_n)$, get $W_n \subset W_{n+1} \subset \ldots \subset W_{n+k}$ and apply the above lemma. Then $(T - \lambda_n \operatorname{id})W_n = W_{n-1}$ because $(T - \lambda_n \operatorname{id})(x_n) = 0$; $(T - \lambda_n \operatorname{id})(x_m) = (\lambda_m - \lambda_n)x_m$ for $m < n$. Thus $\inf_{n\in\mathbb{N}} |\lambda_n| = \lim_{n\to\infty} |\lambda_n| = 0$.

4. Fix $\lambda \in \operatorname{spec}(T) \setminus \{0\}$, then there exists a sequence of spaces

$$W_1 := \operatorname{Ker}(T - \lambda) \subset W_2 := \operatorname{Ker}((T - \lambda)^2) \subset \ldots \subset W_n := \operatorname{Ker}((T - \lambda)^n)$$
$$(T - \lambda)^n = \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^k T^{n-k} = \underbrace{\sum_{k=0}^{n-1} \binom{n}{k} (-\lambda)^k T^{n-k}}_{\in \mathcal{K}(V)} - \underbrace{\lambda(-\lambda)^{n-1} \operatorname{id}}_{\neq 0}$$

So by Riesz-Schauder $(T - \lambda)^n$ is Fredholm and $\dim W_n < \infty$. That means $V_n := (T - \lambda)^n V$ is a closed subspace of V. $V_1 \supset V_2 \supset \ldots$; $V_n^\perp = \operatorname{Ker}(\underbrace{(T' - \lambda)^n}_{\in \mathcal{F}(V)})$ and so $\dim(V/V_n)$ is finite and increasing. $y \in W_n \Rightarrow x = (T - \lambda)y \in (T - \lambda)W_n \subset W_{n-1}$ because $(T - \lambda)^{n-1}x = (T - \lambda)^n y = 0$. Applying again the above lemma where $\inf_{n\in\mathbb{N}} |\lambda_n| = 0$ does not hold, we get $\exists p \in \mathbb{N} \forall n \in \mathbb{N}\colon W_p = W_{p+n}$. Let $p \in \mathbb{N}$ be minimal with this property, then define $N(\lambda) := W_p$.

This argument applies to T' in the same way and so there is some $q \in \mathbb{N}$ with $\operatorname{Ker}(V_q) := \operatorname{Ker}(T' - \lambda \operatorname{id}_{V'})^q = \operatorname{Ker}(T' - \lambda \operatorname{id}_{V'})^{q+n}$ for all $n \in \mathbb{N}$. Then we define $R(\lambda) := V_q$. So $(T-\lambda)|_{R(\lambda)}$ has a trivial kernel and is Fredholm.

We shall show now that $W_q + V_q = V$. Let $x \in V \Rightarrow (T-\lambda)^q x \in V_q = V_{2q} \Rightarrow \exists y \in V\colon S^q x = S^{2q} y \Leftrightarrow (x - S^r y) \in W_q \Rightarrow x = \underbrace{(x - S^q y)}_{\in W_q} + \underbrace{S^q y}_{V_q}$.

Suppose $p > q$:

$$V_q = V_p \wedge \exists x\colon x \in W_p \setminus W_q$$
$$V = W_q + V_q \Rightarrow \exists y \in W_q\colon z \in V_q = V_p \wedge x = y + z$$
$$\Rightarrow z = x - y \in W_p + W_q = W_p (\Leftarrow W_q \subset W_p)$$
$$z \in V_p \Rightarrow z \in W_p \cap V_p = \{0\} \Leftrightarrow z = 0 \Rightarrow x = y \in W_q \quad \lightning$$

Suppose $p < q$:

$$W_p = W_q \wedge \exists x \in V_p \setminus V_q\colon x = y + zy \in W_q, z \in V_q \Rightarrow y = x - z \in V_p + V_q = V_p$$
$$\Rightarrow y \in W_p \cap V_p = \{0\} \Rightarrow x = z \in V_q \quad \lightning$$

$TN(\lambda) \subset N(\lambda), TR(\lambda) \subset R(\lambda)$ because $T = (T-\lambda) + \lambda$ and $(T-\lambda)R(\lambda) = V_{q+1} = V_r = R(\lambda)$.

$N(\lambda) + R(\lambda) = V$, $N(\lambda) \cap R(\lambda) = \{0\}$, $\dim(N(\lambda)) < \infty$, $R(\lambda)$ closed.

$(T-\lambda)R(\lambda) = R(\lambda)$ implies that $(T-\lambda)|_{R(\lambda)}$ is open. Let $(T-\lambda)x = 0$ and $x \in R(\lambda) \Rightarrow x \in W_1 \cap R(\lambda) \subset W_p \cap V_p = \{0\} \Rightarrow x = 0 \Rightarrow (T-\lambda)|_{R(\lambda)}$ is invertible in $\mathcal{B}(R(\lambda))$.

$(T-\lambda)^p(N(\lambda)) = \{0\}$ and $p \in \mathbb{N}$ is the smallest with this property by our definition of p and $N(\lambda) = W_p$.

Proof (114) Let $w \in W_n \subset W_{n+1}$, then $\lambda_n w - Tw \in W_n$ by the precondition. Thus $Tw \in (-W_n + \lambda_n w) = W_n$ for any $w \in W_n$.

In the case that there exists some $n_0 \in \mathbb{N}$ with $W_{n_0} \neq W_{n_0-1}$, there is some $x_{n_0} \in W_{n_0}$ with $\|x_{n_0}\| \leq 2$ and $\|p_{W_{n_0}-1}(x_{n_0})\| = 0$, because $W_{n_0}/W_{n_0-1} \neq \{0\}$ and $\pi_{W_{n_0}-1}(U(0,2)) \supset K(0,1)$. Since $Tx_0 - \lambda_n x_0 \in W_{n_0-1}$ it follows that $\operatorname{dist}(Tx_0, W_{n_0-1}) = \operatorname{dist}(\lambda_{n_0} x_0, W_{n_0-1}) = |\lambda_{n_0}|$

Take $f \in C([0,1] \times [0,1])$, $g \in C([0,1])$ and

$$T(g)(s) := \int_0^s f(s,t) g(t) \mathrm{d}t$$

Then T is compact, $\operatorname{spec} T = \{0\}$, so all $\lambda - T$ are invertible although the equation $Tg = 0$ and $Tg = h$ are difficult to solve.

12 Hilbert spaces

115 **Definition: "Pre-Hilbert Space"**

A normed space $(V, \|\cdot\|)$ that satisfies the parallelogram equation:

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

is called a *pre-Hilbert space*. See Definition 7 (Banach Space, Hilbert Space).

116 **Definition: "Scalar Product"**
A map $\langle\cdot,\cdot\rangle\colon V\times V\to\mathbb{K}$ is called *scalar product* if it satisfies:

1. $\langle x_1+\alpha x_2, y\rangle = \langle x_1, y\rangle + \alpha\langle x_2, y\rangle$
2. $\langle y, x\rangle = \overline{\langle x, y\rangle}$
3. $\langle x, x\rangle > 0 \Leftarrow x \neq 0$

117 **Lemma: Scalar Product on Hilbert Spaces**
If $(V,\langle\cdot,\cdot\rangle)$ is a vector space with a scalar product then $(V,\|\cdot\|)$ is pre-Hilbert w.r.t. $\|v\|:=\sqrt{\langle v,v\rangle}$.
Is on the other hand $(V,\|\cdot\|)$ a given pre-Hilbert space, then

$$\langle x,y\rangle_{\mathbb{R}} := \frac{1}{4}(\|x+y\|^2-\|x-y\|^2) = \tfrac{1}{2}(\|x+y\|^2-\|x\|^2-\|y\|^2)$$

is a real scalar product. If V is complex then

$$\langle x,y\rangle := \langle x,y\rangle_{\mathbb{R}} - i\langle ix,y\rangle_{\mathbb{R}} = \frac{1}{4}\sum_{j=0}^{3} i^k\|x+i^k y\|$$

as $\langle ix,y\rangle_{\mathbb{R}} := -\langle x,iy\rangle_{\mathbb{R}}$.

118 **Lemma: Properties of Hilbert Spaces**
Let H be a Hilbert space

1. H is uniformly convex, with the estimation function $\varphi(t) := \sqrt{4-(2-t)^2}$.
2. So H is reflexive.
3. (Riesz-Fréchet-Fischer Theorem) The map $\Phi_H\colon H\to H'$ with $\Phi_H(v)(y) := \langle y,v\rangle$ is an isometric, surjective, conjugate-linear map from H onto H'.
4. For $T\in\mathcal{B}(H_1,H_2)$ we can define $T^* := \Phi_{H_1}^{-1}\circ T'\circ\Phi_{H_2}$, called the *adjoint* of T, via
$$\forall x\in H_2, y\in H_1\colon \langle T^*x,y\rangle_{H_1} = \langle x,Ty\rangle_{H_2}$$

119 **Definition: "Normal, Self-Adjoint, Unitary and Projection Operators"**
$T\in\mathcal{B}(H)$ is called

normal iff $T^*T = TT^*$

self-adjoint iff $T^* = T$

unitary iff $T^*T = \mathrm{id} = TT^*$

projection iff $T^* = T = T^2$

120 **Lemma: Cauchy-Schwarz-Inequality**

If $\langle\cdot,\cdot\rangle$ is a positiv-definit hermitian sesquilinearform on V then

$$\forall v, w \in V\colon |\langle v, w\rangle|^2 \leq \langle v, v\rangle\langle w, w\rangle$$

121 **Remark: Pre-Hilbert Completion**

The triangle inequality with respect to $\|\cdot\| := \sqrt{\langle\cdot,\cdot\rangle}$ is called the *Minkowski inequality*.

If a pre-Hilbert space $(V, \langle\cdot,\cdot\rangle)$ is given, we can form the completion $(\hat{V}, \|\cdot\|)$ of the corresponding vector space $(V, \|\cdot\|)$ and then the corresponding pre-Hilbert space $(\hat{V}, \langle\cdot,\cdot\rangle)$ is already complete as the function $(v, w) \mapsto \|v+w\|^2 + \|v-w\|^2 - 2(\|v\|^2 + \|w\|^2)$ is continuous and vanishes on V which is dense in $\hat{V}$.

Proof (Riesz-Fréchet-Fischer) According to Cauchy-Schwarz-Inequality we get

$$|\Phi_H(w)(v)| = |\langle v, w\rangle| \leq \|v\|\|w\|$$
$$\Rightarrow \|\Phi_H(w)\| \leq \|w\| \Rightarrow \|\Phi\| \leq 1$$

Also $\Phi(0) = 0$ and for $w_0 := \|w\|^{-1}w, w \neq 0$:

$$\Phi(w)(w_0) = \langle w_0, w\rangle = \|w\|^{-1}\|w\|^2 = \|w\|$$
$$\Rightarrow \|\Phi(w)\| \geq \|w\|$$

So Φ is isometric. Conjugate-linearity follows now:

$$\Phi(\alpha w_1 + w_2)(v) = \langle v, \alpha w_1 + w_2\rangle = \overline{\langle \alpha w_1 + w_2, v\rangle}$$
$$= \overline{\alpha}\overline{\langle w_1, v\rangle} + \overline{\langle w_2, v\rangle} = \overline{\alpha}\langle v, w_1\rangle + \langle v, w_1\rangle$$
$$\Rightarrow \Phi(\alpha w_1, w_2) = \overline{\alpha}\Phi(w_1) + \Phi(w_2)$$

So $\Phi(H)$ is a closed linear subspace of H'. On the other hand it is norming for H: For each $v \in H \setminus \{0\}$ and $v_0 := \|v\|^{-1}v$ we get $\Phi(v_0)(v) = \|v\|$ and $\|\Phi(v_0)\| = \|v_0\| = 1$.

Now assume that $\Phi(H) \subsetneq H'$. Then by Hahn-Banach-Separation there exists a linear functional $F\colon H' \to \mathbb{K}$ with $F \neq 0$ and $F(\Phi(H)) = \{0\}$. And by reflexivity of H we find $v \in H$ with $e_H(v) = F$

$$\forall w \in H\colon 0 = F(\Phi(w)) = e_H(v)(\Phi(w)) = \Phi(w)(v) = \langle v, w\rangle$$
$$\Rightarrow 0 = \langle v, v\rangle = \|v\|^2 \lightning$$

Thus Φ is surjectiv.

122 **Lemma: Distance**

Let H be a Hilbert space, $M \subset H$ a closed convex subset of H, $v_0 \in H$ and $m_0 \in M$. Then the following properties of m_0 and v_0 are equivalent:

1. $\|v_0 - m_0\| = \inf_{m\in M} \|v_0 - m\| = \operatorname{dist}(v_0, M)$
2. $\forall m \in M\colon \operatorname{Re}\langle v_0 - m_0, m - m_0\rangle \leq 0$

Proof (122) For $t \in (0,1)$ look at

$$\|v_0 - (tm_0 + (1-t)m)\|^2 = \|v_0 - m_0\|^2 + (1-t)\left((1-t)\|m - m_0\|^2 - 2\operatorname{Re}\langle v_0 - m_0, m - m_0\rangle\right)$$

123 **Remark: Orthogonal Distance**
In the case where M is a closed linear subspace of H we get:

$$\|v_0 - m_0\| = \operatorname{dist}(v_0, M) \Leftrightarrow \forall m \in M : \langle v_0 - m_0, m\rangle = 0$$

In other words, the shortest connection between a point v_0 and M is orthogonal to M.

124 **Definition: "Orthogonality"**
Let H be a pre-Hilbert space and $X, Y \subset H$. Then X and Y are called *orthogonal* or $X \perp Y$ iff

$$\forall x \in X, y \in Y : \langle x, y\rangle = 0$$

125 **Theorem: Orthogonal Projections onto Closed Linear Subspace**
Let H be a Hilbert space, $M \subset H$ a closed convex subset and $v_0 \in H$, then

1. There is a unique $m_0 \in M$ with $\|v_0 - m_0\| = \operatorname{dist}(v_0, M)$.
2. If M is a linear subspace of H, $P_M(v_0) := m_0 \in M$ is the unique vector with $(v_0 - m_0) \perp M$.
3. $P_M : H \to M$ is a linear map and $P_M \circ P_M = P_M$.

126 **Remark: Closedness of Orthogonal Complement**
If $X \subset H$ is any subset, the set $X^\perp := \{\, x \in H \mid v \perp X \,\}$ is a closed linear subspace of H, $(X^\perp)^\perp = \overline{\operatorname{span}(X)}$ and $\overline{\operatorname{span}(X)}^\perp = X^\perp$ by Hahn-Banach-Separation: Suppose $x_0 \in (X^\perp)^\perp \setminus \overline{\operatorname{span}(X)}$. Then there is some $f \in H'$ which is equal to $\Phi(w)$, with $f|_{\overline{\operatorname{span}(X)}} = 0$ and $f(x_0) \neq 0$ so that $\forall x \in X : \langle x, w\rangle = 0$ ↯.

127 **Definition: "Orthogonal Projector"**
If L is a closed linear subspace of a Hilbert space H, then the linear operator P_L is called the *orthogonal projector* from H onto L.
In General a *orthogonal projection* is some $P \in \mathcal{B}(H)$ with $P^2 = P$ and $\|P\| \leq 1$.

128 **Lemma: Orthogonal Projections**
Let H be a Hilbert space and $P \in \mathcal{B}(H)$ with $P \circ P = P \neq 0$, then the following conditions are equivalent:

1. P is an orthogonal projection
2. $\forall x, y \in H : \langle Px, y\rangle = \langle x, Py\rangle$ (P is self-adjoint)

3. $\forall x \in H\colon \langle Px, x\rangle \geq 0$ and P is self-adjoint
4. $\|P\| = 1$
5. $\forall x, y \in H\colon \|Px - x\| \leq \|Py - x\|$
6. $\|Px - x\|^2 + \|Px\|^2 = \|x\|^2$
7. P is normal

129 **Lemma: Properties of the Adjoint**

Let there be Hilbert spaces H_1 and H_2 as well as $S, T \in \mathcal{B}(H_1, H_2)$, $R \in \mathcal{B}(H_2, H_3)$ and $\lambda \in \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then

1. $(\lambda S + T)^* = \overline{\lambda} S^* + T^*$
2. $(R \circ S)^* = S^* \circ R^*$
3. $S^* \in \mathcal{B}(H_2, H_1)$ and $\|S^*\| = \|S\|$
4. $(S^*)^* = S$
5. $\|SS^*\| = \|S^*S\| = \|S\|^2$. This is called the C^*-Property of the operator-norm on $\mathcal{B}(H)$.
6. $\mathrm{Ker}(S) = \mathrm{Im}(S^*)^\perp$ and $\mathrm{Ker}(S^*) = \mathrm{Im}(S)^\perp$.
7. $\|T\| = \sup_{\substack{\|x\|\leq 1 \\ \|y\|\leq 1}} |\langle Tx, y\rangle| = \|T'\|$ for $x \in H_1, y \in H_2$

130 **Lemma: Characterisation of Self-Adjoint Operators**

Let $T\colon H_1 \to H_2$ be any map (may not be linear) between Hilbert spaces H_1 and H_2 so that there is some $S\colon H_2 \to H_1$ with $\forall x \in H_1, y \in H_2\colon \langle Tx, y\rangle = \langle x, Sy\rangle$, then $T \in \mathcal{B}(H_1, H_2)$ and $S = T^*$.

Proof (130)

$$\begin{aligned}
&\langle Tx_1 + \alpha T_2, y\rangle = \langle Tx_1, y\rangle + \alpha\langle Tx_2, y\rangle = \langle x_1, Sy\rangle + \alpha\langle x_2, Sy\rangle \\
&= \langle x_1 + \alpha x_2, Sy\rangle = \langle T(x_1 + \alpha x_2), y\rangle \\
&\Rightarrow (Tx_1 + \alpha Tx_2) - T(x_1 + \alpha x_2) \perp H_2 \Rightarrow Tx_1 + \alpha Tx_2 = T(x_1 + \alpha x_2)
\end{aligned}$$

We show, that the graph of T is closed, to see that T is continuous:

$$\begin{aligned}
&(x, y) \in \mathrm{Graph}(T) \Leftrightarrow \forall z \in H\colon \langle Tx, z\rangle = \langle x, Sz\rangle = \langle y, z\rangle \\
&\|(x, y)\|_2 := \sqrt{\|x\|_{H_1}^2 + \|y\|_{H_2}^2}
\end{aligned}$$

Then $H_1 \oplus_2 H_2$ is a Hilbertspace with the scalarproduct $\langle (x_1, z_1), (x_2, z_2)\rangle := \langle x_1, x_2\rangle + \langle z_1, z_2\rangle$.

$$\Rightarrow \mathrm{Graph}(T) = \left\{ (Sz, -z) \;\middle|\; z \in H_2 \right\}^\perp$$

Thus Graph(T) is closed and according to Theorem 60 (Closed Graph Theorem) T is continuous.

131 **Lemma: Self-Adjointness Condition**
Let $\mathbb{K} = \mathbb{C}$ and $T \in \mathcal{B}(H)$, then the following conditions are equivalent:

1. T is self-adjoint, $T = T^*$
2. $\forall x \in H\colon \langle Tx, x\rangle \in \mathbb{R}$

Proof (131)

(i) ⇒ (ii) $\langle Tx, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle} \Rightarrow \langle Tx, x\rangle \in \mathbb{R}$

(ii) ⇒ (i) For any complex value λ:

$$\langle T(x+\lambda y), (x+\lambda y)\rangle = \langle Tx, x\rangle + \overline{\lambda}\langle Tx, y\rangle + \lambda\langle Ty, x\rangle + |\lambda|^2\langle Ty, y\rangle$$
$$\Rightarrow \forall\lambda \in \mathbb{C}\colon \overline{\lambda}\langle Tx, y\rangle + \lambda\langle Ty, x\rangle \in \mathbb{R}$$
$$\Rightarrow \langle Tx, y\rangle = \overline{\langle Ty, x\rangle} \Rightarrow \langle Tx, y\rangle \in \mathbb{R}$$

Counterexample for $\mathbb{K} = \mathbb{R}$ Let $H = \mathbb{R}^2$ and look at the rotation

$$T\begin{pmatrix}\alpha\\ \beta\end{pmatrix} := \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

then $\forall x \in H\colon \langle Tx, x\rangle = 0$ and $T^* = -T$.

132 **Theorem: Norm of Self-Adjoint Operators**
Let $T \in \mathcal{B}(H)$ be self-adjoint and $\dim H > 0$, then

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|$$

Proof (132) Let $M := \sup_{\|x\|=1} |\langle Tx, x\rangle|$.

$$|\langle Tx, x\rangle| \le \|Tx\|\|x\| \le \|T\|\|x\|^2 \le \|T\| \Leftarrow \|x\| = 1$$
$$\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle = 2(\langle Tx, y\rangle + \langle Ty, x\rangle) = 4\operatorname{Re}(\langle Tx, y\rangle)$$
$$\le M(\|x+y\|^2 + \|x-y\|^2) \le 2M(\|x\|^2 + \|y\|^2) \le 4M \Leftarrow \|x\|, \|y\| \le 1$$
$$\Rightarrow \operatorname{Re}\langle Tx, y\rangle \le M \Rightarrow |\operatorname{Re}\langle Tx, y\rangle| \le M$$

With $\varphi \in \mathbb{R}$ such that $e^{i\varphi}\langle Tx, y\rangle = |\langle Tx, y\rangle|$ and $y' := e^{-i\varphi}y$ we get

$$|\langle Tx, y\rangle| = \operatorname{Re}\langle Tx, e^{-i\varphi}\rangle \le M$$

133 **Corollary: Norm Equation**
For $T \in \mathcal{B}(H)$ with $T = T^*$ and $\langle Tx, x\rangle \ge 0$ the equation $\|2T - \|T\|\,\mathrm{id}_H\| = \|T\|$ holds, because the self-adjoint operators form a closed, $\mathbb{R}$-linear subspace of $\mathcal{B}(H)$.

134 **Remark: Norm of Normal Operators**

Even for any normal operator $T \in \mathcal{B}(H)$ we have

$$\|T\| = \sup_{\|x\|\leq 1} |\langle Tx, x\rangle|$$

135 **Corollary: Properties of Operators**

Let H be a Hilbert space, then

1. $\forall x \in H\colon \langle Tx, x\rangle = 0$ implies $T = 0$ for all self-adjoint $T \in \mathcal{B}(H)$
2. and for all $T \in \mathcal{B}(H)$ if H is a complex space. In that case we also have

$$\|T\| \leq 2 \sup_{\|x\|\leq 1} |\langle Tx, x\rangle|$$

Proof (135) We can decompose T into self-adjoint operators:

$$T = \tfrac{1}{2}(T^* + T) + i(\frac{1}{2i}(T - T^*)) =: T_1 + iT_2$$
$$\Rightarrow M \geq |\langle Tx, x\rangle| = |\langle (T_1 + iT_2)x, x\rangle| = |\langle T_1 x, x\rangle + i\langle T_2 x, x\rangle|$$
$$= \sqrt{|\langle T_1 x, x\rangle|^2 + |\langle T_2 x, x\rangle|^2} \geq \max(|\langle T_1 x, x\rangle|, |\langle T_2 x, x\rangle|)$$
$$\|T_1\| \leq M \wedge \|T_2\| \leq M \Rightarrow \|T\| = \|T_1 + iT_2\| \leq \|T_1\| + \|T_2\| \leq 2M$$

136 **Remark:**

1. Let $e_1, \dots, e_n$ be pairwise orthogonal, $\|e_k\| = 1$ for $k = 1, \dots, n$, then the projection on the linear subspace $L := \operatorname{span}\{e_1, \dots, e_n\}$ is $P_L(x) = \sum_{k=1}^n \langle x, e_k\rangle e_k$.
2. $S \in \mathcal{B}(H)$ then S is normal iff $\forall x \in H\colon \|Sx\| = \|S^* x\|$
3. $\operatorname{spec}(T^*) = \overline{\operatorname{spec}(T)}$

137 **Definition: "Unconditional Convergence"**

Let $\{x_i\}_{i\in I}$ be a family of vectors $x_i \in V$ in a normed space V. We say that $\sum_{i\in I} x_i$ *converges unconditional* to $y \in V$ if for every $\varepsilon > 0$ there is a finit subset $J \subset I$ such that for any finit set K, $J \subset K \subset I$ we have

$$\|y - \sum_{i\in K} x_i\| < \varepsilon$$

For unconditional convergent series

$$\sum_{i\in I} (x_i + \alpha y_i) = \sum_{i\in I} x_i + \alpha \sum_{i\in I} y_i$$
$$I = K \sqcup J \Rightarrow \sum_{i\in I} x_i = \sum_{i\in K} x_i + \sum_{i\in J} x_i$$

Example Let $(\alpha_1, \alpha_2, \dots) \in \ell_2$ and $I = \mathbb{N}$ then $\sum_{i \in \mathbb{N}} \alpha_i e_i$ is conditional convergent for $\alpha \in \ell_2$ and absolutely convergent for $\alpha \in \ell_1 \subsetneq \ell_2$. (e.g. $\alpha_n = \frac{1}{n}$)

138 **Definition: "Orthonormal System"**
Let H be Hilbert, then $X \subset H$ is called *orthonormal system* (ONS) if

$$\forall x \in X \colon \|x\| = 1$$
$$\forall x, y \in X \colon x \perp y$$

An ONS is called *maximal* if there is no genuine superset which is a ONS. We also say *complete ONS* or *orthonormal basis* (ONB). (This need not be a vector space basis!)

139 **Theorem: Properties of Orthonormal Systems**
Let H be Hilbert, $S \subset H$ an ONS, then

1. There exists an ONB $S' \supset S$.
2. Bessels Inequation holds:
$$\forall x \in H \colon \sum_{e \in s} |\langle x, e \rangle|^2 \leq \|x\|$$
3. Parseval-Equation
$$\sum_{e \in s} |\langle x, e \rangle|^2 = \|x\|$$
$$x = \sum_{e \in S} \langle x, e \rangle e$$
iff $x \in \overline{\text{span}(S)}$.
4. The orthogonal projection onto $L = \overline{\text{span}(S)}$ is given by
$$P_L(x) = \sum_{e \in S} \langle x, e \rangle e$$
5. The following conditions on S are equivalent:
   a) S is an ONB
   b) $S^\perp = \{0\}$
   c) $\forall x \in H \colon \sum_{e \in S} \langle x, e \rangle e = x$
   d) $H = \overline{\text{span}(S)}$
   e) $\forall x \in H \colon \|x\|^2 = \sum_{e \in S} |\langle x, e \rangle|^2$
   f) $\forall x, y \in H \colon \langle x, y \rangle = \sum_{e \in S} \langle x, e \rangle \overline{\langle y, e \rangle}$

140 **Lemma: Properties of Normal Operators**

Let T be normal, then there is a $z \in \operatorname{spec} T$ with $|z| = \|T\| = r(T)$. This can be seen with the C^*-property of the norm:

$$\|T^2\|^2 = \|(T^2)^*T^2\| = \|(T^*T)^2\| = \|T^*T\|^2 = \|T\|^4$$
$$\Rightarrow \|T^{2k}\| = \|T\|^{2k}$$
$$\Rightarrow r(T) = \|T\|$$

More generally we can also say, that every polynomial over a normal operator is normal again.

141 **Lemma: Decomposition of the Spectrum**

Let $T \in \mathcal{B}(H)$, then $\lambda \in \operatorname{spec} T$ is equivalent to exactly one of the following conditions:

1. $T - \lambda$ is not injective, i.e. λ is an eigenvalue of T
2. $T - \lambda$ is injective and the image is not dense in H. This is equivalent to $\operatorname{Ker}(T^* - \bar{\lambda}) = \operatorname{Ker}((T - \lambda)^*) \neq \{0\}$ and thus to the condition, that $\bar{\lambda}$ is an eigenvalue of T^*.
3. $T - \lambda$ is injective, $T^* - \bar{\lambda}$ is injective but $T - \lambda$ is not surjective. Thus there must be a sequence $(x_n), x_n \in H, \|x_n\| = 1$ so that $\lim_{n\to\infty} \|(T - \lambda)x_n\| = 0$

142 **Remark: Self-Adjoint Operators**

Is T self-adjoint, this characterization yields a proof for $\operatorname{spec} T \subset \mathbb{R}$. In the first two cases we get for $\lambda \in \operatorname{spec} T$:

$$\lambda = \langle \lambda x, x\rangle = \langle Tx, x\rangle = \langle x, T^*x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle} \Rightarrow \lambda \in \mathbb{R}$$

and in the third:

$$|\langle Tx_n, x_n\rangle - \lambda| = |\langle (Tx_n - \lambda x_n), x_n\rangle| \leq \|Tx_n - \lambda x_n\|\|x_n\| \to 0$$

143 **Corollary: Spectrum of Self-Adjoint Operators**

1. If T is self-adjoint, $\operatorname{spec} T \subset \mathbb{R}$ is compact and for $\alpha = \min \operatorname{spec} T$ and $\beta = \max \operatorname{spec} T$ the equation $\|T\| = \max\{|\alpha|, |\beta|\}$ holds.
2. $\alpha = \inf \langle Tx, x\rangle$, $\beta = \sup \langle Tx, x\rangle$, $x \in H$, $\|x\| = 1$

144 **Remark:**

1. Suppose that $\lambda x = Tx$ and T is normal then $\bar{\lambda} x = T^* x$, because $T - \lambda$ is normal and $T - \lambda = T^* - \bar{\lambda}$ so we have

   $$\|(T - \lambda)x\|^2 = \|(T - \lambda)^* x\|^2 = \|T^* - \bar{\lambda}\|^2$$

2. Eigenvectors for different eigenvalues are orthogonal.
3. If T is normal and $T^n x = 0$, then already $Tx = 0$.

145 **Theorem:**
Let $T \in \mathcal{K}(H)$ be normal then there exists an orthonormal basis S of H such that

$$Tx = \sum_{e \in S} f(e)\langle x, e\rangle e$$

for some $f \in c_0(S) = \left\{ f \colon S \to \mathbb{C} \;\middle|\; |f|^{-1}(t, \infty) \text{ finite for all } t > 0 \right\}$